Name: Solutions

Directions: Show all work. No credit for answers without work. You may leave your answers in terms of factorials and, when necessary, sums with a few terms.

1. **[3 parts, 2 points each]** How many ways are there to arrange the letters of 'TRIATHLETE':

   (a) with no additional restrictions?

   T: 3
   R: 1
   I: 1
   A: 1
   H: 1
   L: 1
   E: 2

   Symbols: 10.

   So total is $\boxed{\dfrac{(10)!}{(3)!\,(2)!}} = \boxed{302{,}400}$

   (b) that start with at least two T's? (For example, both 'TTRIAHLETE' and 'TTTRIAHLEE' count.)

   \# such arrangements = \# arrangements of $\underbrace{\text{TRIAHLEE}}$ ← 8 symbols, 2E's

   $\underbrace{\text{TT}}$RAHEIELT ⟷ RAHEIELT

   So \# is $\boxed{\dfrac{8!}{2!}} = \boxed{20{,}160}$

   (c) that have the 'A' between the two 'E's (not necessarily consecutively)? (For example, both 'TRIETHLATE' and 'TRIEAETHLT' count.)

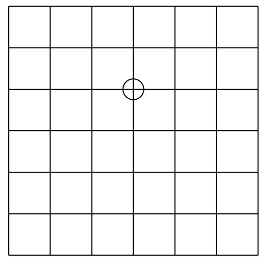
   \# such arrangements = \#arrangements of $\underbrace{\text{TRIZTHLZTZ}}$

   10 symbols
   3 Z's
   3 T's

   TRIETHLATE ⟷ TRIZTHLZTZ

   So \# is $\boxed{\dfrac{10!}{(3!)(3!)}} = \boxed{100{,}800}$

2. **[2 points]** Recall that each step in a *lattice path* increases one of the coordinates by 1. Out of all lattice paths from $(0,0)$ to $(6,6)$, determine the fraction that pass through the point $(3,4)$.



Total # paths = # arrangements of $\underbrace{R\cdots R}_{6}\ \underbrace{U\cdots U}_{6}$ $= \dfrac{(12)!}{(6!)(6!)} = 924$

Total # paths $(0,0)$ to $(3,4)$ = # arrangements of RRRUUUU $= \dfrac{7!}{(3!)(4!)} = 35$

Total # paths $(3,4)$ to $(6,6)$ = # arrangements of RRRUU $= \dfrac{5!}{3!\cdot 2!} = 10$

Total # paths $(0,0)$ to $(6,6)$ through $(3,4)$ $= \dfrac{7!\cdot 5!}{(3!)(4)!\cdot(3)!\cdot(2)!} = 35\cdot 10 = 350$

So fraction through $(3,4)$ is $\boxed{\dfrac{350}{924}}$ or $\boxed{\dfrac{25}{66}}$, approx 37.88%

3. **[2 points]** Suppose that $n \geq 2$. How many ways are there to arrange the integers in $\{1,\ldots,n\}$ so that 1 and n are not next to each other? For example, if $n=5$, then 2 3 5 4 1 counts but 2 3 5 1 4 does not.

Count the complement.

Total # arrangements $= P(n,n) = n!$

Total # arrangements with 1 and n consecutive:

1. Arrange $\overbrace{2,3,4,\cdots,n-2,n-1,*}^{n-1 \text{ distinct symbols}}$ $\quad n_1 = (n-1)!$
2. Replace $*$ with $1,n$ or $n,1$ $\quad n_2 = 2$

$2(n-1)!$

So total # is $\boxed{n! - 2(n-1)!} = n\cdot(n-1)! - 2(n-1)! = \boxed{(n-2)(n-1)!}$