Directions: Solve the following problems. See the course syllabus and the Homework Webpage on the course website for general directions and guidelines.

1. Let p be an odd prime. Determine the number of mutually incongruent solutions to $x^2+y^2 \equiv 0 \pmod{p}$. (A solution (x,y) is congruent to (x',y') if $(x,y) \equiv (x',y') \pmod{p}$. When $p=3$, there is 1 solution $(0,0)$, and when $p=5$, there are 9 solutions.)

2. *Infinitely many primes congruent to* 7 *modulo* 8.

   (a) Prove that if n is an integer and p is an odd prime dividing n^2-2, then $p \equiv \pm 1 \pmod{8}$.

   (b) Prove that there are infinitely many primes p such that $p \equiv 7 \pmod{8}$.

3. Sums of three squares.

   (a) [NT 11-2.9] Show that no integer of the form $4^a(8m+7)$ is the sum of three squares. Hint: consider the congruence $x^2+y^2+z^2 \equiv 7 \pmod{8}$.

   (b) Prove or disprove: if x and y are representable as the sum of three squares, then so is xy.

   (c) Prove that if x is representable as the sum of three *positive* squares, then so is x^2.

4. Partition Exercises.

   (a) Find the conjugate partition to $16 = 5+4+4+2+1$.

   (b) [NT 12-3.1] For the case $n=8$, list the corresponding pairs of partitions of n in which all parts are odd and partitions of n into distinct parts given by Theorem 12-3.

5. **[Challenge]** Prove or disprove: there are infinitely many integer pairs (a,b) such that

$$2a^2 - b^2 = 1.$$

6. **[Challenge]** Let p be a prime with $p \equiv 1 \pmod{4}$. Count the number of solutions to $x_1^2 + \cdots + x_n^2 \equiv 0 \pmod{p}$. (The case $n=2$ appears in problem 1.)