Directions: Solve the following problems; challenge problems are optional for extra credit. See the course syllabus and the Homework Webpage on the course website for general directions and guidelines.

1. [NT 2-3.1] Find the general solution (if solutions exist) of each of the following linear Diophantine equations:

   (a) $15x + 51y = 41$

   (b) $23x + 29y = 25$

   (c) $121x - 88y = 572$

2. Binomial Coefficients and Parity.

   (a) [NT 3-1.3] Using the definition of $\binom{n}{r}$, show combinatorially that $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$. (To show an identity combinatorially, find an appropriate set A and show that both sides of the identity count the elements in A.)

   (b) Prove that if n is even and r is odd, then $\binom{n}{r}$ is even.

3. Prove that n^5 and n have the same last digit.

4. Prove that if a and b are positive integers, then it is not possible for both $a + b^2$ and $a^2 + b$ to be square numbers (i.e. of the form k^2 for some integer k). Hint: after a^2, what is the next largest square?

5. Prove that if p is an odd prime, then there are infinitely many integers n such that $p \mid n2^n + 1$.

6. Prove that if n is an integer and $n \geq 2$, then $n^4 + 4^n$ is not prime.

7. **[Challenge]** Fermat's "medium" theorem?

   (a) Let p and q be distinct primes. Count the number of cyclic lists of length pq with entries in a set of size n. (For example, for $p = 2$, $q = 3$, and $n = 2$, we are counting cyclic lists of length 6 with entries in, say, $\{\text{red}, \text{blue}\}$; there are 14 of these.)

   (b) Use part (a) to show that if p and q are distinct primes and n is a positive integer, then pq divides $n^{pq} - n^p - n^q + n$.