Name: Solutions

Directions: Show all work. Answers without work generally do not earn points.

1. **[4 parts, 3 points each]** Let $\Sigma = \{0, 1, 2\}$. Define the following languages.

   $A = \{w \mid$ the symbols of w are in sorted order (non-decreasing from left to right)$\}$
   $B = \{w \mid w$ has more 1's than 0's$\}$
   $C = \{w \mid$ the symbols of w can be split into two groups whose sums are equal$\}$

   For example: λ and 01201 are both words in C (we can split 01201 into two groups with equal sum by putting both one's in the first group and the remaining symbols in the second group), and $1211 \notin C$ (no split is possible).

   (a) Give an example of a string in $A \cap B \cap C$.

   11, 011112

   (b) Give an example of a string in $A - B$.

   001, 0011

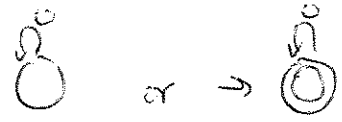
   (c) True or False: $A \cap B \subseteq C$.

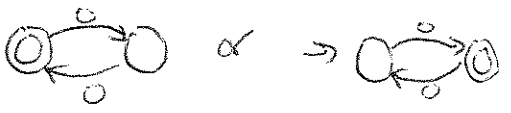
   False: $1 \in A \cap B$ but $1 \notin C$.

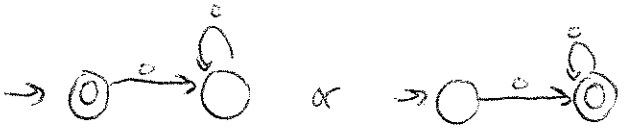
   (d) True or False: $BB = B$.

   FALSE: $101 \in B$ but $101 \notin BB$.

2. **[4 points]** Let $\Sigma = \{0\}$. Find all languages over Σ that are computable by DFAs with at most 2 states.

   One state:  giving $\emptyset$ and Σ^*

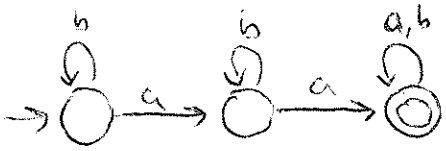
   Two states:  giving $\{w : w$ has even length$\}$ and $\{w : w$ has odd length$\}$

    giving $\{\lambda\}$ and $\{0^n : n \geq 1\}$

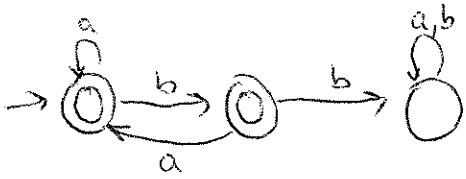
   So there are 6 languages: $\emptyset$, Σ^*, $\{w : |w| \text{ is even}\}$, $\{w : |w| \text{ is odd}\}$, $\{\lambda\}$, $\{0^n : n \geq 1\}$

3. **[4 parts, 4 points each]** Let $\Sigma = \{a, b\}$. Construct DFAs for the following languages.

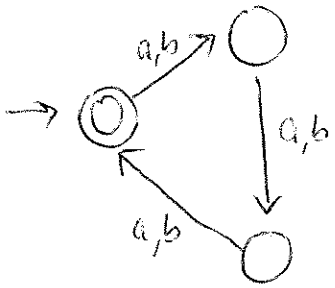
(a) $\{w \mid w$ has at least two a's$\}$



(b) $\{w \mid w$ has no pair of consecutive b's$\}$



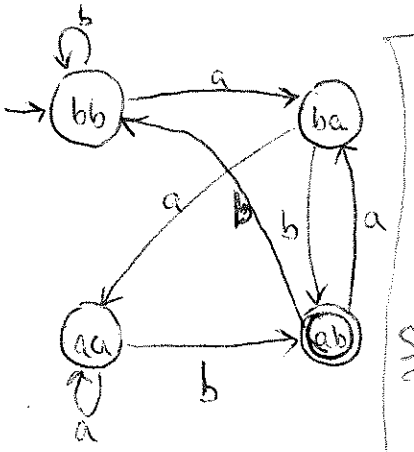
(c) $\{w \mid$ the length of w is divisible by 3$\}$



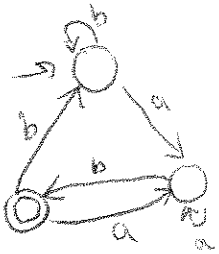
(d) $\{w \mid w$ ends with $ab\}$

Keep track of last two symbols.

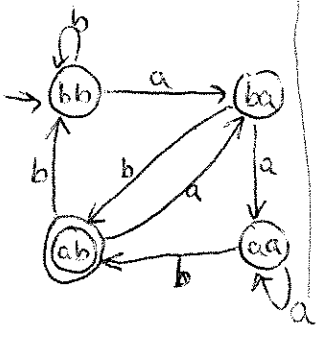
Appropriate start state: bb



Simplified.



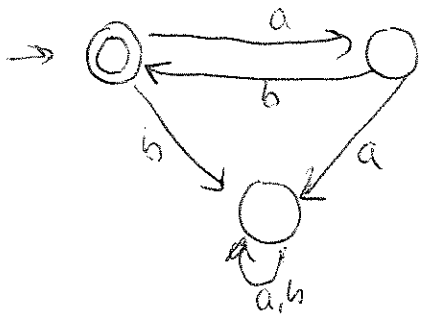
Redrawn:



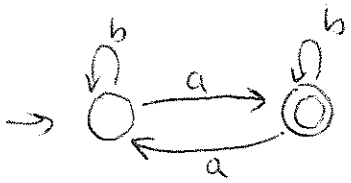
4. Let $\Sigma = \{a, b\}$ and define the following languages.

$$A_1 = \{(ab)^n \mid n \geq 0\} = \{\lambda, ab, abab, ababab, \ldots\}$$
$$A_2 = \{w \mid w \text{ has an odd number of } a\text{'s}\}$$

(a) **[5 points]** Give a simplified DFA for A_1.



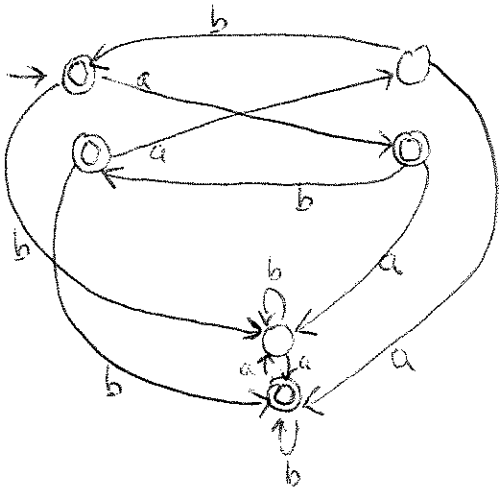
(b) **[5 points]** Give a simplified DFA for A_2.



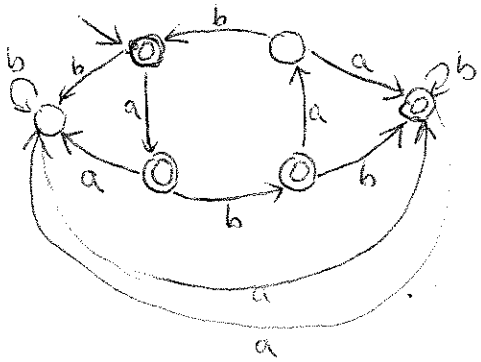
(c) **[8 points]** Give a DFA for $A_1 \cup A_2$.

Use product construction.

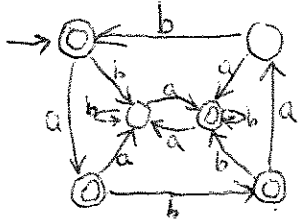
O even # a's
O odd # a's



Redrawn:



Better drawing:

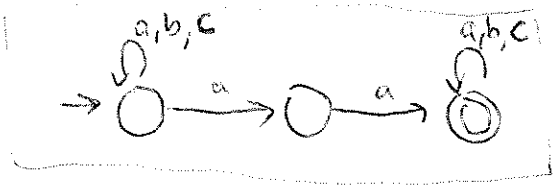


5. **[4 parts, 4 points each]** Let $\Sigma = \{a, b, c\}$. For each string $w \in \Sigma^*$, let $\#a(w)$, $\#b(w)$, and $\#c(w)$ denote the number of a's, b's, and c's in w. Define the following languages.

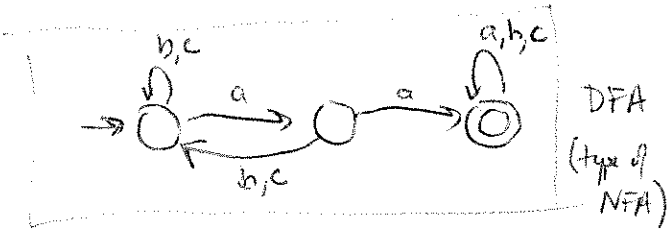
$A = \{w \mid w \text{ contains a consecutive pair of } a\text{'s}\}$
$B = \{w \mid \#b(w) = 1 \text{ and the single } b \text{ in } w \text{ is the second to last symbol}\}$
$C = \{w \mid \text{the 3-tuple } (\#a(w), \#b(w), \#c(w)) \text{ contains at least one even integer}\}$

Give NFAs for the following using at most the specified number of states.

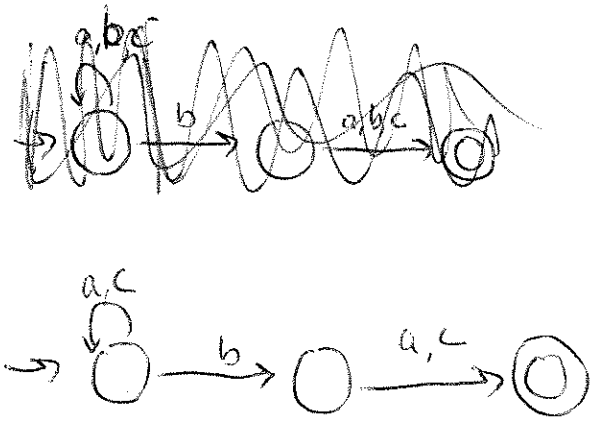
(a) A, at most 3 states



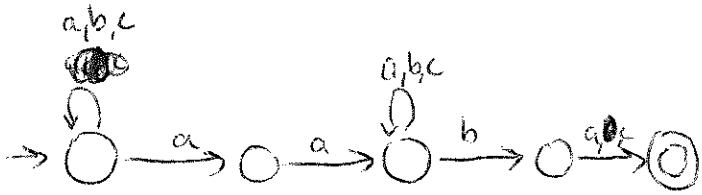
or



(b) B, at most 3 states



(c) AB, at most 5 states

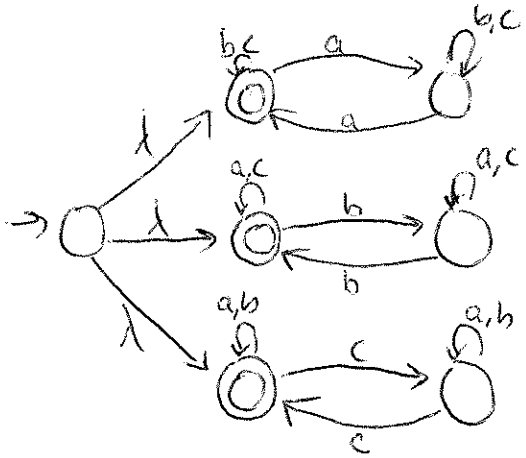


IDEA: NFA splits word into

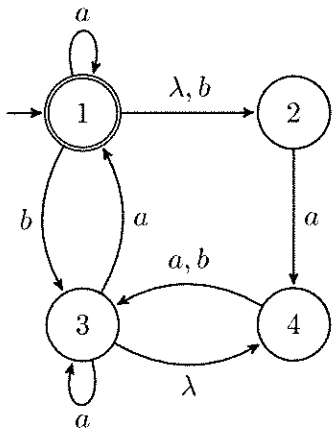
$x \in A$ and $y \in B$, where $|y| = 2$

(d) C, at most 7 states

IDEA: Guess whether the a's, b's, c's will be even and check.



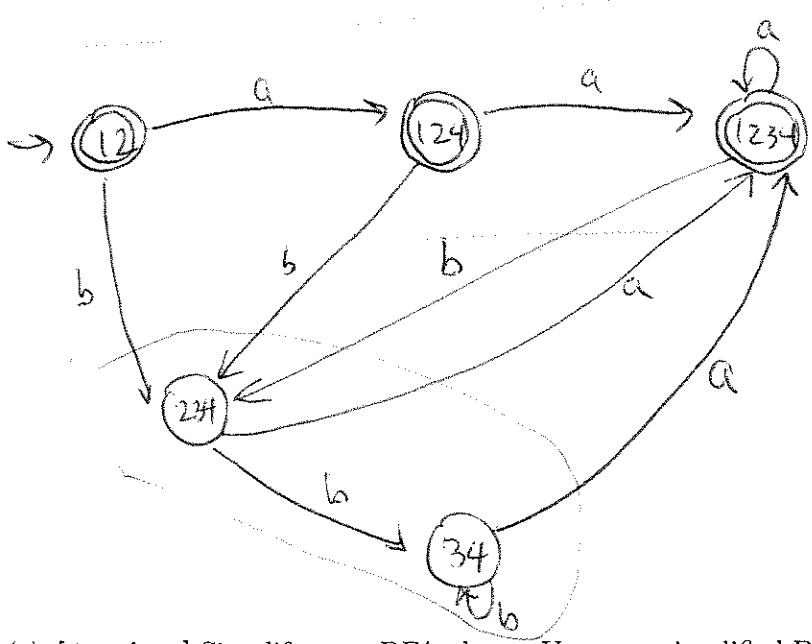
6. Let $\Sigma = \{a, b\}$ and let N be the following NFA (below left).



State	a	b
1	1,2,4	2,3,4
2	4	∅
3	1,2,3,4	3,4
4	3,4	3,4.

(a) **[8 points]** For each state/input pair, give the set of successor states in the table above.

(b) **[6 points]** Convert N to a DFA.



(c) **[4 points]** Simplify your DFA above. Use your simplified DFA to give a simple description of the language computed by N.

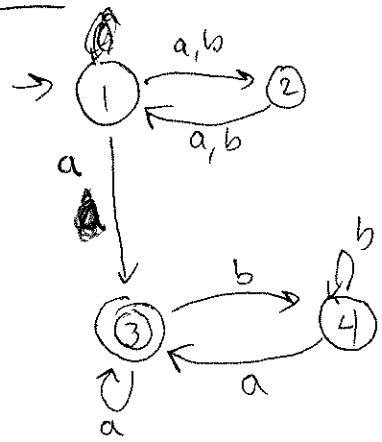
$L(N) = \{w : w \text{ does not end in a } b\}$

7. **[16 points]** Let $\Sigma = \{a, b\}$ and define the following languages.

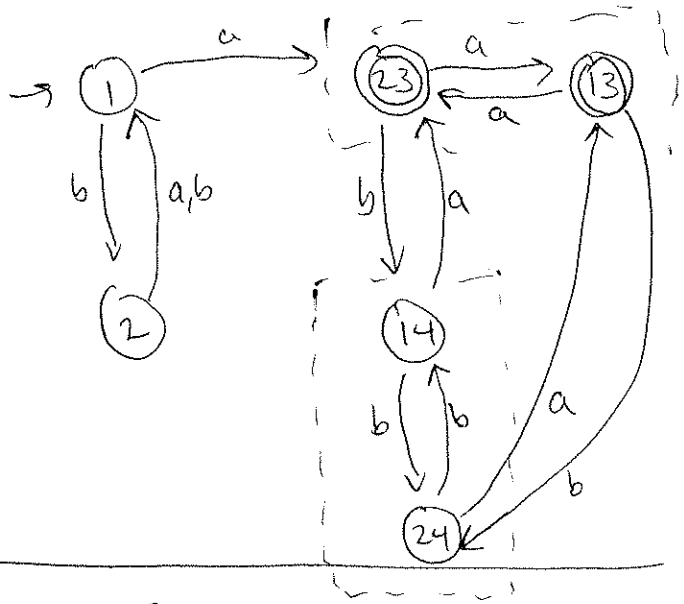
$$A = \{w \mid w \text{ starts and ends with an } a\}$$
$$B = \{w \mid w \text{ has an even length}\}$$

Note that $\lambda \notin A$ but $a \in A$. Give a ~~simplified~~ DFA that computes the language BA.

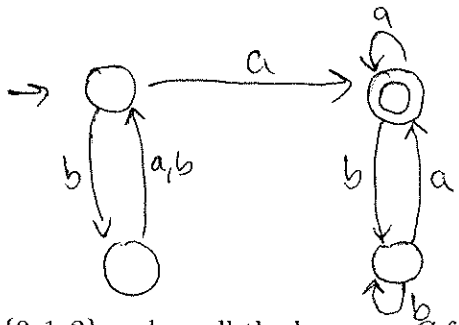
NFA: for BA



DFA:



Simplified (optional):



8. **[4 bonus points]** Let $\Sigma = \{0, 1, 2\}$ and recall the language C from problem (1), where

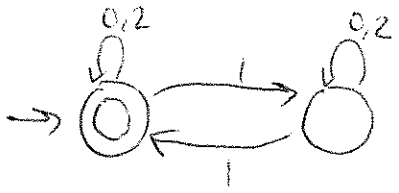
$$C = \{w \mid \text{the symbols of } w \text{ can be split into two groups whose sums are equal}\}.$$

Is the language C regular? Justify your answer.

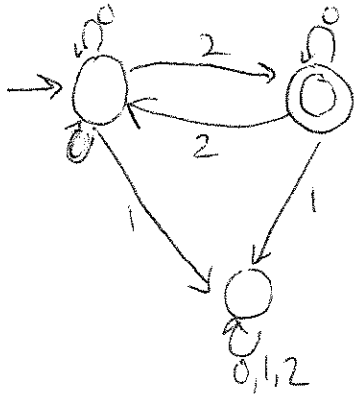
Regular. Let $C_1 = \{w \mid$ sum of symbols in w is even$\}$ and let $C_2 = \{w \mid w \in \{0,2\}^* \text{ and } w \text{ has an odd number of 2's}\}$. One can show that $C = C_1 - C_2$. Since C_1 and C_2 are regular, by closure properties of DFAs, C is also regular.

Scratch Paper

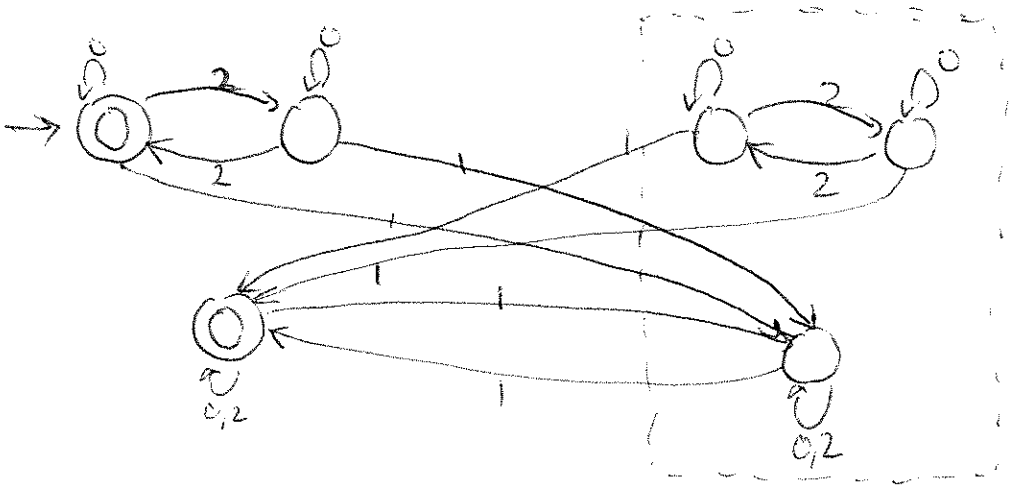
A DFA for C_1:



A DFA for C_2:



A DFA for C, $C=C_1-C_2$:



Simplify: