Name: Solution Key

Directions: Show all work. Answers without work generally do not earn points. Unless stated otherwise, answers may be left in terms of factorials and binomial coefficients.

1. **[4 points]** How many subsets of $\{1, 2, \ldots, 14\}$ have size 10? Express your answer as a concrete, simplified number.

$$\binom{14}{10} = \frac{(14)!}{(10)!\,4!} = \frac{14\cdot13\cdot12\cdot11}{4\cdot3\cdot2\cdot1} = \boxed{1001}$$

2. **[3 parts, 4 points each]** A restaurant offers a combo meal; customers choose one of 5 sandwiches, one of 3 sides, and one of 7 beverages. Charles and Marie wish to order combo meals.

   (a) How many ways can Charles and Marie order combo meals?

   # combo meals $= 5\cdot3\cdot7 = 105$

   $$\underbrace{105}_{\text{Charles}} \cdot \underbrace{105}_{\text{Marie}} = \boxed{(105)^2} = \boxed{11{,}205}$$

   (b) How many ways can Charles and Marie order so that they avoid placing identical orders? (For example, a situation in which Charles and Marie order the same sandwich, the same beverage, but different sides counts.)

   $$\boxed{\underbrace{105}_{\text{Charles}} \cdot \underbrace{104}_{\text{Marie, different from Charles}}} = \boxed{10{,}920}$$

   (c) How many ways are there for Charles and Marie to order combo meals with different sandwiches, different sides, and different beverages?

   Stage 1: Charles orders $n_1 = 5\cdot3\cdot7$

   Stage 2: Marie orders, each choice different from Charlie's $n_2 = 4\cdot2\cdot6$

   So # ways is $\boxed{(5\cdot3\cdot7)\cdot(4\cdot2\cdot6)}$ or $\boxed{5{,}040}$

3. **[4 parts, 4 points each]** A 4-digit ATM pin is a list of 4 digits, like 0000 and 5398. How many 4-digit ATM pins:

(a) are there in total?

Stage 1: Choose first digit $n_1=10$

⋮

Stage 4: Choose fourth digit $n_4=10$

$\boxed{10^4} = \boxed{10{,}000}$

(b) do not contain the digit 3? (So 1425 counts but 8322 does not.)

Stage 1: Choose first digit (not 3) $n_1 = 9$

⋮

Stage 4: " fourth " " $n_4 = 9$

$\boxed{9^4} = \boxed{6561}$

(c) contain all distinct digits? (So 5398 counts but 5395 does not.)

$\boxed{P(10,4)} = \boxed{10 \cdot 9 \cdot 8 \cdot 7} = \boxed{5040}$

(d) contain at least one even digit and and least one odd digit? (So 3011 counts but 0284 and 5555 do not.)

Ans = #(pins total) − #(pins with only even digits) − #(pins with only odd digits)

$= 10^4 - 5^4 - 5^4 = \boxed{10^4 - 2\cdot 5^4} = \boxed{8{,}750}$

4. **[4 parts, 4 points each]** How many ways are there to arrange the letters in ~~CORROBORATION~~:

C-1 R-3 A-1 I-1
O-4 B-1 T-1 N-1

(a) with no additional restrictions?

$$\frac{13!}{4!\cdot 3!\cdot (1!)^6} = \boxed{\frac{13!}{4!\cdot 3!}} = \boxed{43,243,200}$$

(b) with all four O's appearing consecutively? (So BROOOORCTAIRR counts but ROOBOORCTAIRR does not.)

Glue "O's and arrange C<OOOO>RRRBAITN:

$$\frac{10!}{3!\,(1!)^7} = \boxed{\frac{10!}{3!}} = \boxed{604,800}$$

(c) with the O's, the A, and the I appearing before all other letters? (So OIOOAO|RCBRTNR counts but OIOOAROCBRTNR does not.)

Stage 1: Arrange OIOOAO $\quad n_1 = \frac{6!}{4!\,(1!)^2} = \frac{6!}{4!} = 6\cdot 5$

Stage 2: Arrange RCBRTNR $\quad n_2 = \frac{7!}{3!\,(1!)^4} = \frac{7!}{3!}$

So total is $\boxed{\frac{6!}{4!}\cdot\frac{7!}{3!}} = \boxed{30\cdot\frac{7!}{3!}} = \boxed{5\cdot 7!} = \boxed{25,200}$

(d) with all O's appearing before all R's? (So COOBOONRRRTAI counts but CORBOONRORTAI does not.)

Stages: Replace both O and R with new letter *, arrange C**B**N***TAI, and then replace the *'s with 4 "O"s and 3 "R"s, in order.

$$\frac{13!}{7!\cdot(1!)^6} = \boxed{\frac{13!}{7!}} = \boxed{1,235,520}$$

5. **[5 parts, 4 points each]** A non-standard card deck has 6 suits (called A, B, C, D, E, and F) and 9 ranks (1 through 9). The deck has one card for each suit/rank pair, for a total of 54 cards. When a variant of poker is played with this deck, a *hand* consists of a set of 7 cards. For example, {2A, 3A, 1B, 8B, 2D, 5E, 9F} is a hand. How many hands:

   (a) are there in total?

   54 cards, choose 7 for hand.

   $$\boxed{\binom{54}{7}} = \boxed{177,100,560}$$

   (b) have all 7 cards in the same suit?

   Stage 1: Choose suit $n_1 = 6$

   Stage 2: Choose 7 cards within suit $n_2 = \binom{9}{7}$

   So $\boxed{6 \cdot \binom{9}{7}} = \boxed{216}$.

   (c) contain no cards of suit A and no cards of rank ~~9~~ 1?

   There are 9 cards with suit A, and 6 cards with rank 1, so a total of $(9+6-1)$ or 14 cards to avoid. (Careful! 1A is just a single card.) Choose 7 cards from remaining 40.

   So $\boxed{\binom{40}{7}} = \boxed{18,643,560}$

   (d) have no two cards with the same rank?

   Stage 1: Choose 7 distinct ranks $n_1 = \binom{9}{7}$

   Stage 2: Pick a suit for each rank $n_2 = \underbrace{6 \cdot 6 \cdots 6}_{7 \text{ copies}}$

   $\boxed{\binom{9}{7} \cdot 6^7} = \boxed{10,077,696}$

   (e) have a majority (i.e. more than half) of cards belonging to suit A? Express your answer using Sigma (Σ) summation notation.

   # hands with k cards in suit $A = \binom{9}{k} \cdot \binom{54-9}{7-k} = \binom{9}{k} \cdot \binom{45}{7-k}$

   from suit A — from other suits

   ~~So~~ To get more than half, add from $k=4$ to $k=7$: $\boxed{\sum_{k=4}^{7} \binom{9}{k}\binom{45}{7-k}}$

   $= \boxed{1,916,496}$

6. **[4 parts, 4 points each]** A pet store offers 6 types of community fish: danios, guppies, swordtails, platies, rasboras, and tetras. Determine the number of ways to purchase:

   (a) 3 fish, with all fish of distinct types? (So "1 guppy, 1 tetra, and 1 platy" counts, but "2 swordtails and 1 danio" does not.)

   $$\boxed{\binom{6}{3}} = \boxed{20}$$

   (b) 3 fish, with no additional restrictions? (So "3 tetras" counts.)

   ~~Use~~ Count # solns to $x_1 + \cdots + x_6 = 3$.

   Three stars, 5 bars. |||*|* *|

   So $\binom{5+3}{3} = \boxed{\binom{8}{3}} = \boxed{56}$

   (c) 15 fish, with at least one fish of each available type?

   Count # solns to $x_1 + \cdots + x_6 = 15$ with each $x_i \geq 1$.

   So Count # solns to $x_1 + \cdots + x_6 = 9$ with each $x_i \geq 0$.

   $\Rightarrow$ 9 stars, 5 bars

   $\Rightarrow \binom{9+5}{5} = \boxed{\binom{14}{5}} = \boxed{2002}$.

   (d) ~~between~~ at least 10 ~~and~~ and at most 20 fish?

At most 20 fish: # solns to $x_1 + \cdots + x_6 + y = 20$.

20 stars, 6 bars $\Rightarrow \binom{26}{6}$.

At most 9 fish: # solns to $x_1 + \cdots + x_6 + y = 9$

9 stars, 6 bars $\Rightarrow \binom{15}{6}$

So ans $= \boxed{\binom{26}{6} - \binom{15}{6}} = \boxed{225,225}$

7. **[3 parts, 4 points each]** Determine the following coefficients.

(a) The coefficient of x^4y^2 in $(x+y)^6$.

$$\binom{6}{4} x^4 y^2 \Rightarrow \boxed{\binom{6}{4}} = \boxed{15}$$

(b) The coefficient of x^8 in $(x+2)^{20}$.

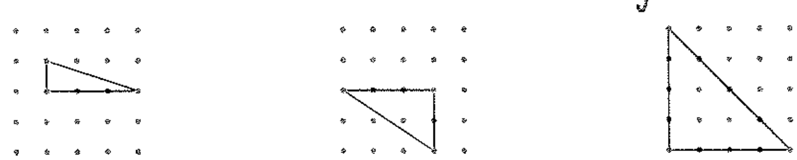
$$\binom{20}{8} x^8 \cdot 2^{20-8} = \binom{20}{8} \cdot 2^{12} x^6$$

$$\text{So } \boxed{\binom{20}{8} \cdot 2^{12}} = \boxed{515{,}973{,}120}$$

(c) The coefficient of $w^3x^6y^2z$ in $(w+x+y+z)^{12}$.

$$\text{Multinomial } \boxed{\frac{12!}{3!\,6!\,2!\,1!}} = \boxed{55{,}440}$$

8. **[4 points]** A *right triangle* is a triangle having an angle of 90 degrees. How many right triangles can be formed whose vertices belong to a set of 25 points arranged in a 5×5 grid? ~~Three examples are shown below.~~ With one horizontal leg and one vertical leg?



Stage 1: Choose vertex (x,y) for the right angle $n_1 = 25$

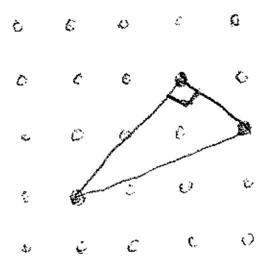
Stage 2: Choose a second vertex in same row as (x,y) $n_2 = 4$

Stage 3: Choose a third vertex in same column as (x,y) $n_3 = 4$.

So $\boxed{25 \cdot 4 \cdot 4} = \boxed{400}$.

Scratch Paper

Note on #8: Solving the problem as originally asked is tricky, because we must also include triangles like



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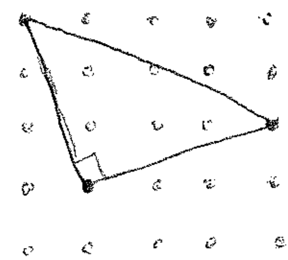
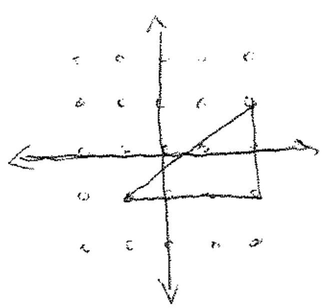
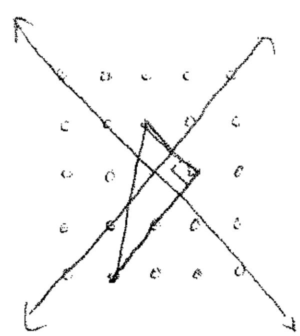


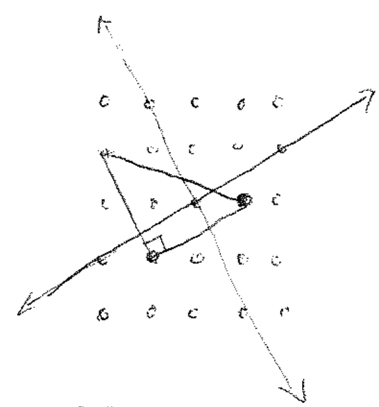
One way to count is to split the right triangles according to the slope of the legs. For example:



400 axis-aligned triangles



124 triangles with axis slopes 1, −1



28 triangles with axis slopes $\frac{1}{2}$, −2

There 596 ~~for~~ right triangles after adding all the possibilities