Solutions

1. A population of ants grows logistically. Initially, the ant population is 10% of the carrying capacity. After 1 year, the ant population has doubled. Compute
   (a) The population (as a percentage of carrying capacity) after 2 years.
   (b) The time at which the population reaches 90% of carrying capacity.
   (c) The time at which the population is increasing fastest.

   Hint: recall the logistic equation $\frac{dy}{dt} = r(1-(y/K))y$. Let $y(t)$ be the population of ants in units of carrying capacity, so that $K = 1$, $y(0) = 0.1$, and $y(1) = 0.2$.

(a)

$$\frac{dy}{dt} = r(1-y)y$$

$$\int \frac{1}{y(1-y)}\,dy = \int r\,dt$$

PARTIAL FRACTIONS:

$$\frac{1}{y(1-y)} = \frac{A}{y} + \frac{B}{1-y}$$

$$1 = A(1-y) + By$$

$$1 = A + (B-A)y$$

$\Rightarrow A = 1$

$B - A = 0 \Rightarrow B = 1.$

$$\int \frac{1}{y} + \frac{1}{1-y}\,dy = \int r\,dt$$

$$\ln|y| - \ln|1-y| = rt + C$$

$$\ln\left(\frac{y}{1-y}\right) = rt + C$$

$$\frac{y}{1-y} = Ce^{rt}$$

Impose $y(0) = 0.1$:

$$\frac{0.1}{1-0.1} = C$$

$$C = \frac{\frac{1}{10}}{\frac{9}{10}} = \frac{1}{9}$$

$$\frac{y}{1-y} = \frac{1}{9}e^{rt}$$

Impose $y(1) = 0.2$:

$$\frac{0.2}{1-0.2} = \frac{1}{9}e^{r\cdot 1}$$

$$\frac{\frac{1}{5}}{\frac{4}{5}} = \frac{1}{9}e^{r}$$

$$e^{r} = \frac{9}{4}$$

$$\frac{y}{1-y} = \frac{1}{9}(e^{r})^{t}$$

$$\frac{y}{1-y} = \frac{1}{9}\left(\frac{9}{4}\right)^{t}$$

$$9y = \left(\frac{9}{4}\right)^{t}(1-y)$$

$$\left(9 + \left(\frac{9}{4}\right)^{t}\right)y = \left(\frac{9}{4}\right)^{t}$$

$$\boxed{y = \frac{(9/4)^{t}}{9 + \left(\frac{9}{4}\right)^{t}}}$$

(a) $y(2) = \dfrac{(9/4)^2}{9 + \left(\frac{9}{4}\right)^2} = \dfrac{9}{25} = 0.36 = \boxed{36\%}$

(b) Solve for t:

$$0.9 = \frac{(9/4)^{t}}{9 + \left(\frac{9}{4}\right)^{t}}$$

$$\frac{9}{10}\left(9 + \left(\frac{9}{4}\right)^{t}\right) = \left(\frac{9}{4}\right)^{t}$$

$$\frac{81}{10} = \left(\frac{9}{4}\right)^{t}\left[1 - \frac{9}{10}\right]$$

$$\left(\frac{9}{4}\right)^{t} = 81$$

$$t\ln\left(\frac{9}{4}\right) = \ln(81)$$

$$t = \boxed{\frac{\ln(81)}{\ln(9/4)} \approx 5.42 \text{ years}}$$

(c) Find when $\frac{dy}{dt} = f(y) = r(1-y)y$ is maximized.

- $f'(y) = \dfrac{df}{dy} = \dfrac{d}{dy}\left[r(1-y)y\right] = r(-y + (1-y))$
  $= r(1-2y).$
- Find critical pts of $f(y)$: $r(1-2y) = 0$
  $y = \frac{1}{2}$
- So $f(y)$ maximized at $\cancel{y=0}$, $y = \frac{1}{2}$, $\cancel{\text{or } y=1}$.
- Find t value for $y = \frac{1}{2}$.
- $\dfrac{1}{2} = \dfrac{(9/4)^{t}}{9 + \left(\frac{9}{4}\right)^{t}} \Longrightarrow t = \dfrac{\ln(9)}{\ln(9/4)} \approx \boxed{2.71 \text{ years}}$

2. Find an integrating factor $\mu(x)$ that depends only on x to solve

$$\frac{dy}{dx} = -\left(\frac{y\sin x + 2yx(\cos x)}{x\sin x}\right).$$

Hint: rewrite the equation in standard differential form. After transforming to an exact equation, try imposing $\psi_y = N$ first.

$$\underbrace{(y\sin x + 2yx\cos x)}_{M} + \underbrace{(x\sin x)}_{N}\frac{dy}{dx} = 0$$

$M_y = \sin x + 2x\cos x$

$N_x = \sin x + x\cos x$

WANT: $(\mu M)_y = (\mu N)_x$, $\mu = \mu(x)$.

$\cancel{\mu_y}^{\,0} M + \mu M_y = \mu_x N + \mu N_x$

$\mu' N = \mu(M_y - N_x)$

$\frac{\mu'}{\mu} = \frac{M_y - N_x}{N}$

$\frac{\mu'}{\mu} = \frac{x\cos x}{x\sin x}$

$\frac{\mu'}{\mu} = \cot x$

$\int \frac{1}{\mu}\,d\mu = \int \cot x\,dx$

$\ln|\mu| = \int \frac{\cos x}{\sin x}\,dx$ $\quad u = \sin x$, $du = \cos x\,dx$

$\ln|\mu| = \ln|\sin x| + C$

$\mu = \sin x$

$(y\sin^2 x + 2yx\cos x\sin x) + (x\sin^2 x)\frac{dy}{dx} = 0.$

• $\psi = \int x\sin^2 x\,dy = xy\sin^2 x + g(x)$

• Impose $\psi_x = M$:

$\frac{\partial}{\partial x}\left[xy\sin^2(x) + g(x)\right] = y\sin^2 x + 2yx\cos x\sin x$

$\cancel{y\sin^2(x) + yx\cdot 2\sin(x)\cdot\cos(x)} + g'(x) = \cancel{y\sin^2 x + 2yx\cos(x)\sin(x)}$

$g'(x) = 0$

$g(x) = \int 0\,dx = C$

So $\psi = xy\sin^2 x$,

~~Gen~~ soln; implicit:

$xy\sin^2 x = C$

Explicit:

$$\boxed{y = \frac{C}{x\sin^2 x}}$$

3. Compute the following.

(a) $\frac{3+2i}{4-i}$

$= \frac{(3+2i)(4+i)}{(4-i)(4+i)} = \frac{12 + 11i + 2i^2}{16 - i^2}$

$= \frac{10 + 11i}{17} = \boxed{\frac{10}{17} + \frac{11}{17}i}$

(b) $(2+i)e^{1-\frac{\pi}{2}i} = (2+i)\left[e\cdot e^{(-\frac{\pi}{2})i}\right]$

$= (2+i)\cdot\left[e\cdot\left(\cos(-\tfrac{\pi}{2}) + i\sin(-\tfrac{\pi}{2})\right)\right]$

$= (2+i)e\left[0 + i(-1)\right]$

$= e(2+i)(-i)$

$= e(-2i - i^2) = e(-2i + 1)$

$= \boxed{e - 2ei}$