1. Give qualitative analysis of the following autonomous differential equations. That is, determine the equilibrium solutions, classify each as stable, unstable, or semistable, and sketch the solutions. Include a phase line.

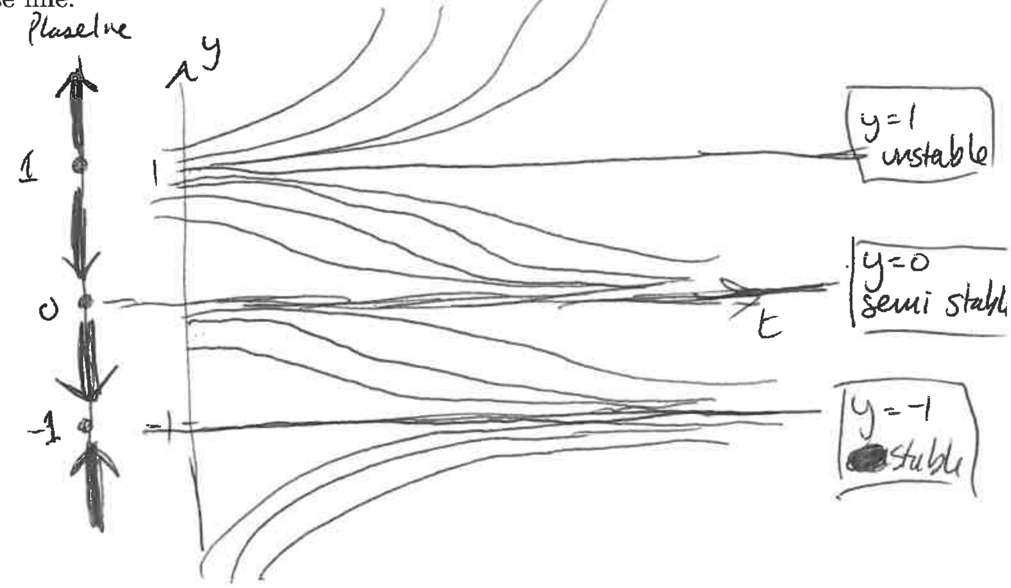
   (a) $\frac{dy}{dt} = y^2(y^2-1)$

Crit pts:

$0 = y^2(y^2-1)$

$0 = y^2(y-1)(y+1)$

$y=0$ or $y=1$ or $y=-1$.




   (b) $\frac{dy}{dt} = \sin y$

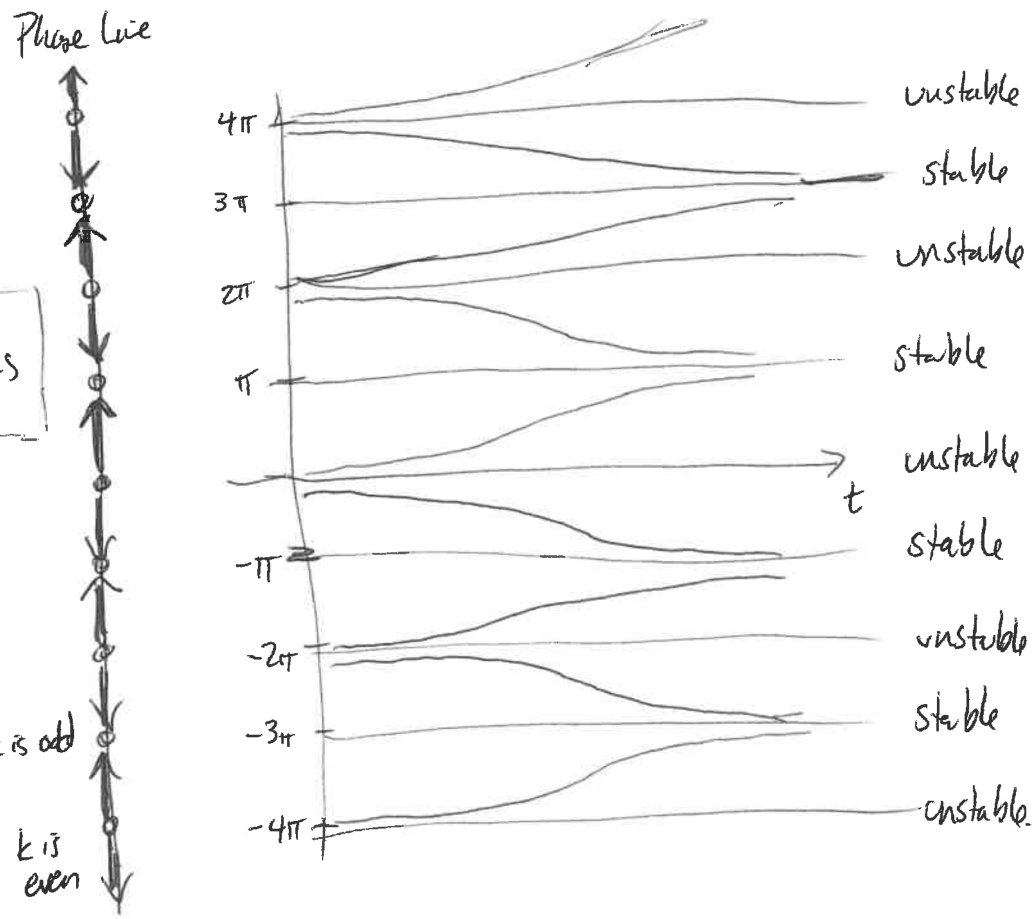
Crit pts:

$0 = \sin y$

$\boxed{y = k\pi, \ k \text{ is an integer. Eq. Solns}}$

Note:

Equilibrium $y = k\pi$ is $\begin{cases} \text{stable if } k \text{ is odd} \\ \text{unstable if } k \text{ is even} \end{cases}$



2. Find an implicit general solution to the following exact equation: $\underbrace{(2xy+\cos x)}_{M}+\underbrace{(x^2+4y)}_{N}y' = 0$.

$N_x = 2x$; $M_y = 2x$ ✓ Exact.

① Impose $\psi_y = x^2+4y$

$\psi = \int x^2+4y\,dy$

$= x^2y + 2y^2 + g(x)$.

② Impose $\psi_x = 2xy + \cos x$

$\frac{\partial}{\partial x}\left[x^2y+2y^2+g(x)\right] = 2xy + \cos x$

$\cancel{2xy} + 0 + g'(x) = \cancel{2xy} + \cos x$

$g'(x) = \cos(x)$

$g(x) = \int \cos(x)\,dx = \sin(x) + C$

③ So $\psi = x^2y + 2y^2 + g(x)$

$= x^2y + 2y^2 + \sin(x) + C$

And gen. soln is

$$\boxed{x^2y + 2y^2 + \sin(x) = C}$$

3. Find an appropriate integrating factor $\mu$ for $\underbrace{(5x^4y+4xy^2)}_{M} + \underbrace{(3x^5+8x^2y)}_{N}y' = 0$ and solve. (Hint: look for $\mu = \mu(y)$.)

WANT: $(\mu M)_y = (\mu N)_x$

$\mu_y M + \mu M_y = \underbrace{\cancel{\mu_x N}}_{0,\text{ since } \mu=\mu(y)} + \mu N_x$

$\mu_y = \mu'$, since $\mu = \mu(y)$.

$\mu' = \frac{\mu(N_x - M_y)}{M}$

$\frac{\mu'}{\mu} = \frac{N_x - M_y}{M}$

- $N_x = \frac{\partial}{\partial x}(3x^5+8x^2y) = 15x^4 + 16xy$
- $M_y = \frac{\partial}{\partial y}\left[5x^4y + 4xy^2\right] = 5x^4 + 8xy$
- $\frac{\mu'}{\mu} = \frac{N_x - M_y}{M} = \frac{10x^4+8xy}{5x^4y+4xy^2} = \frac{2}{y}\cdot\frac{\cancel{5x^4+4xy}}{\cancel{5x^4+4xy}}$

- $\int \frac{\mu'}{\mu}\,dy = \int \frac{2}{y}\,dy$

$\int \frac{1}{\mu}\,d\mu = 2\ln|y| + C$ ← Choose $C=0$

$\ln|\mu| = \ln|y^2|$

$\boxed{\mu = y^2}$ ← Integrating factor

Solve: $(5x^4y^3 + 4xy^4) + (3x^5y^2 + 8x^2y^3)y' = 0$

① Impose

$\psi_x = 5x^4y^3 + 4xy^4$

$\psi = \int 5x^4y^3 + 4xy^4\,dx$

$= x^5y^3 + 2x^2y^4 + g(y)$

② Impose $\psi_y = 3x^5y^2 + 8x^2y^3$:

$\cancel{3x^5y^2} + \cancel{8x^2y^3} + g'(y) = \cancel{3x^5y^2} + \cancel{8x^2y^3}$

$g'(y) = 0$

$g(y) = \int 0\,dy = C$.

③ So $\psi = x^5y^3 + 2x^2y^4 + C$

and general soln is

$$\boxed{x^5y^3 + 2x^2y^4 = C}$$