Solutions

1. [2.2.{14,20}] Solve the following IVPs explicitly. Determine the interval of validity.

   (a) $y' = xy^3(1+x^2)^{-1/2}$ with $y(0) = 1$

$$\frac{y'}{y^3} = \frac{x}{\sqrt{1+x^2}}, \quad \underline{y \neq 0.}$$

$$\int y^{-3}\,dy = \int x(1+x^2)^{-\frac{1}{2}}\,dx \qquad u = 1+x^2, \; du = 2x\,dx$$

$$-\frac{1}{2}y^{-2} = \frac{1}{2}\int u^{-\frac{1}{2}}\,du$$

$$-\frac{1}{2}y^{-2} = u^{\frac{1}{2}} + C$$

$$-\frac{1}{2}y^{-2} = (1+x^2)^{\frac{1}{2}} + C$$

Impose $y(0)=1$:

$$-\frac{1}{2} = 1^{\frac{1}{2}} + C; \quad C = -\frac{3}{2}$$

$$\frac{1}{y^2} = 3 - 2\sqrt{1+x^2}$$

$$y^2 = \frac{1}{3 - 2\sqrt{1+x^2}}$$

$$y = \pm\left(3 - 2\sqrt{1+x^2}\right)^{-\frac{1}{2}}$$

Choose + to satisfy initial condition.

$$\boxed{y = \left(3 - 2\sqrt{1+x^2}\right)^{-\frac{1}{2}}, \quad \text{valid for } |x| < \sqrt{5}/2.}$$

$3 - 2\sqrt{1+x^2} > 0$, $\sqrt{1+x^2} < \frac{3}{2}$ | $1+x^2 < \frac{9}{4}$, $x^2 < \frac{5}{4}$ | $|x| < \frac{\sqrt{5}}{2}$.

   (b) $y^2(1-x^2)^{1/2}\,dy = \arcsin x\,dx$ with $y(0) = 1$.

$$y^2\,dy = \frac{\operatorname{asin} x}{\sqrt{1-x^2}}\,dx, \quad \text{valid for } |x| < 1$$

$$\int y^2\,dy = \int \frac{\operatorname{asin}(x)}{\sqrt{1-x^2}}\,dx \qquad u = \operatorname{asin}(x), \; du = \frac{1}{\sqrt{1-x^2}}\,dx$$

$$\frac{1}{3}y^3 = \int u\,du$$

$$\frac{1}{3}y^3 = \frac{1}{2}u^2 + c$$

$$y^3 = \frac{3}{2}(\operatorname{asin}(x))^2 + C$$

Impose $y(0)=1$:

$$1 = \frac{3}{2}\cdot 0^2 + C; \quad C = 1.$$

$$y = \left[\frac{3}{2}(\operatorname{asin}(x))^2 + 1\right]^{\frac{1}{3}},$$

valid for $|x| < 1$

2. [2.1.28] ~~Consider~~ Solve the IVP $y' + \frac{2}{3}y = 1 - \frac{1}{2}t$ with $y(0) = y_0$. ~~Find the value of y_0 for which the solution touches, but does not cross, the t axis.~~

$$\mu = e^{\int \frac{2}{3} dt} = e^{\frac{2}{3}t}$$

$$e^{\frac{2}{3}t} y' + \frac{2}{3} e^{\frac{2}{3}t} y = \left(1 - \tfrac{t}{2}\right) e^{\frac{2}{3}t}$$

$$\frac{d}{dt}\left[e^{(2/3)t} y\right] = \left(1 - \frac{t}{2}\right) e^{\frac{2}{3}t}$$

$$e^{(2/3)t} y = \int \left(1 - \frac{t}{2}\right) e^{\frac{2}{3}t} dt$$

$$e^{(2/3)t} y = \frac{3}{2} e^{\frac{2}{3}t} - \int \frac{t}{2} e^{(2/3)t} dt \qquad u = \frac{t}{2} \quad v = \frac{3}{2} e^{(2/3)t} \qquad du = \frac{1}{2}dt \quad dv = e^{(2/3)t} dt$$

$$e^{(2/3)t} y = \frac{3}{2} e^{(2/3)t} - \left(\frac{3t}{4} e^{(2/3)t} - \int \frac{1}{2} e^{(2/3)t} dt\right)$$

$$e^{(2/3)t} y = \frac{3}{2} e^{(2/3)t} - \frac{3t}{4} e^{(2/3)t} + \frac{3}{4} e^{(2/3)t} + C$$

Impose $y(0) = y_0$: $1 \cdot y_0 = \frac{3}{2} \cdot 1 - 0 + \frac{3}{4} + C$

$$C = y_0 - \frac{9}{4}$$

$$y = \frac{9}{4} - \frac{3t}{4} + \left(y_0 - \frac{9}{4}\right) e^{-(2/3)t}$$

3. Find the general solution to $y' = \frac{x^2+y^2}{xy}$.

Homogeneous:

$$y' = \frac{\frac{1}{x^2} \cdot (x^2 + y^2)}{\frac{1}{x^2} \cdot xy} = \frac{1 + \left(\frac{y}{x}\right)^2}{\frac{y}{x}}.$$

Let $v = \frac{y}{x}$, so $y = vx$ and

$$\frac{dy}{dx} = \frac{dv}{dx} \cdot x + v.$$

$$\frac{dv}{dx} \cdot x + v = \frac{1 + v^2}{v}$$

$$v' x = \frac{1+v^2}{v} - \frac{v^2}{v}$$

$$v' x = \frac{1}{v}$$

$$v \cdot v' = \frac{1}{x}$$

$$\int v \, dv = \int \frac{1}{x} dx$$

$$\frac{v^2}{2} = \ln|x| + C$$

$$v^2 = 2\ln|x| + C$$

$$v = \pm\sqrt{\ln(x^2) + C}$$

$$\frac{y}{x} = \pm\sqrt{\ln(x^2) + C}$$

$$\boxed{y = \pm x\sqrt{\ln(x^2) + C}.}$$