Directions: Solve 5 of the following 6 problems. All written work must be your own, using only permitted sources. See the "General Guidelines and Advice" on the homework page for more details.

1. Let x and y be vertices in a 3-connected graph G. Show that there is an induced xy-path P such that $G - V(P)$ is connected.

2. Let G be a $2k$-edge-connected graph with at most two vertices of odd degree. Prove that G has a k-edge-connected orientation. (Recall that a digraph D is k-edge-connected if $|[S, \overline{S}]| \geq k$ when S is a nonempty proper subset of $V(D)$. Here, the directed cut $[S, \overline{S}]$ is the set of all edges from vertices in S to vertices outside S.)

3. Minimally k-edge-connected graphs.

   (a) For $S \subseteq V(G)$, let $d(S) = |[S, \overline{S}]|$. Let X and Y be nonempty proper vertex subsets of G. Prove that $d(X \cap Y) + d(X \cup Y) \leq d(X) + d(Y)$. Hint: the sets $X \cap Y$, $X - Y$, $Y - X$, and $\overline{X} \cap \overline{Y}$ partition $V(G)$. Draw a picture in which $V(G)$ is organized by this partition and consider contributions from various types of edges.

   (b) A k-edge-connected graph G is *minimally k-edge-connected* if, for each edge e in G, the graph $G - e$ is not k-edge-connected. Prove that $\delta(G) = k$ when G is minimally k-edge-connected. Hint: Consider a minimal set S such that $|[S, \overline{S}]| = k$. If $|S| \neq 1$, then use $G - e$ for some $e \in E(G[S])$ to obtain another set T with $|[T, \overline{T}]| = k$ such that S, T contradict part (a).

4. Use network flows to prove Menger's Theorem for edge-disjoint paths in graphs: $\kappa'(x, y) = \lambda'(x, y)$. (Recall that $\kappa'(x, y)$ is the minimum size of a set of edges S such that $G - S$ has no xy-path, and $\lambda'(x, y)$ is the maximum size of a set of edge-disjoint xy-paths.)

5. Let G be a graph whose odd cycles are pairwise intersecting, meaning that every two odd cycles in G have a common vertex. Prove that $\chi(G) \leq 5$.

6. Let G be a graph with no induced copy of P_4, let $k = \omega(G)$, and let σ be an ordering of $V(G)$. Prove that with respect to σ, the greedy algorithm produces a proper k-coloring of G. Hint: show that if a vertex u receives color j, then u completes a j-clique with vertices that precede u in σ.