Directions: Solve 5 of the following 6 problems. All written work must be your own, using only permitted sources. See the "General Guidelines and Advice" on the homework page for more details.

1. Let G be a graph with no induced copy of P_4, let $k = \omega(G)$, and let σ be an ordering of $V(G)$. Prove that with respect to σ, the greedy algorithm produces a proper k-coloring of G. Hint: show that if a vertex u receives color j, then u completes a j-clique with vertices that precede u in σ.

2. Prove that a graph G is m-colorable if and only if $\alpha(G \,\square\, K_m) \geq |V(G)|$.

3. *Looseness of* $\chi(G) \geq |V(G)|/\alpha(G)$. Let G be an n-vertex graph.

   (a) Prove that $\chi(G) + \chi(\overline{G}) \leq n + 1$. Hint: use induction on n.

   (b) Let $c = (n+1)/\alpha(G)$. Prove that $\chi(G) \cdot \chi(\overline{G}) \leq (n+1)^2/4$, and use this to prove that $\chi(G) \leq c(n+1)/4$.

   (c) For each odd n, construct a graph G such that $\chi(G) = c(n+1)/4$.

4. Let G be a k-colorable graph, and let P be a set of vertices in G such that $\operatorname{dist}(x, y) \geq 4$ whenever $x, y \in P$. Prove that every coloring of P with colors from $[k+1]$ extends to a proper $(k+1)$-coloring of G.

5. Prove that if G has no induced $2K_2$, then $\chi(G) \leq \binom{\omega(G)+1}{2}$. (Hint: use a maximum clique to define a collection of $\binom{\omega(G)}{2} + \omega(G)$ independent sets that cover the vertices.)

6. Let G be a k-chromatic graph with girth 6 and order n. Construct G' as follows. Let T be an independent set of kn vertices. Take $\binom{kn}{n}$ pairwise disjoint copies of G, one for each way to choose an n-set $S \subset T$. Add a matching between each copy of G and its corresponding n-set S. Prove that the resulting graph has chromatic number $k+1$ and girth 6. (Comment: Since C_6 has chromatic number 2 and girth 6, the process can start and these graphs exist.)