Directions: Solve 5 of the following 6 problems. All written work must be your own, using only permitted sources. See the "General Guidelines and Advice" on the homework page for more details.

1. A *doubly stochastic matrix* Q is a nonnegative real matrix in which every row and every column sums to 1. Prove that a doubly stochastic matrix Q can be expressed as $Q = c_1P_1 + \cdots + c_mP_m$ where $c_1, \ldots, c_m$ are nonnegative real numbers summing to 1 and $P_1, \ldots, P_m$ are permutation matrices. For example,

$$\begin{pmatrix} 1/2 & 1/3 & 1/6 \\ 0 & 1/6 & 5/6 \\ 1/2 & 1/2 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

   Hint: Use induction on the number of nonzero entries in Q.

2. Determine the maximum number of edges in a bipartite graph that contains no matching with k edges and no star with l edges. (Your answer should provide a construction and prove it is best possible.)

3. For even n at least 4, determine the maximum number of edges in a connected n-vertex graph that does not have a perfect matching. (Your answer should provide a construction and prove it is best possible.)

4. For $k > 0$, let G be a k-regular graph of even order that remains connected whenever $k - 2$ edges are deleted. Prove that G has a 1-factor.

5. Prove that a 3-regular graph has a 1-factor if and only if it decomposes into copies of P_4.

6. Connectivity and perfect matchings.

   (a) Let G be an r-connected graph of even order having no $K_{1,r+1}$ as an induced subgraph. Prove that G has a 1-factor.

   (b) For each r, construct an r-connected graph of even order that does not contain an induced copy of $K_{1,r+3}$ and has no 1-factor.

   (Comment: this leaves unresolved whether every r-connected graph of even order without an induced copy of $K_{1,r+2}$ has a 1-factor. Note: when the number of vertices is even, the inclusion bigraph between $(r-1)$-sets and r-sets in $[2r]$ is a candidate for a sharpness example. This graph has no induced K_{r+2} and no perfect matching. Probably it is r-connected.)