Directions: Solve the following problems. See the course syllabus and the Homework Webpage on the course website for general directions and guidelines.

1. Let $(f(n), g(n))$ be a Möbius pair. In class, we showed that if $f(n)$ is multiplicative, then $g(n)$ is multiplicative. Prove that if $g(n)$ is multiplicative, then $f(n)$ is multiplicative. *Comment:* of course, the proof is in the text. Try to do this problem without using the text, using the converse direction as a model.

2. Let M be the set of all positive integers m such that $a^{\phi(m)+1} \equiv a \pmod{m}$ for each integer a. Give a simple characterization of M and prove that your characterization is correct.

3. [NT 5-2.{9,10}]

   (a) Prove that if p is a prime and $p \equiv 1 \pmod 4$, then $\left[\left(\frac{p-1}{2}\right)!\right]^2 \equiv -1 \pmod p$.

   (b) Use the above to find a solution for each of the following.

      i. $x^2 \equiv -1 \pmod{13}$

      ii. $x^2 \equiv -1 \pmod{17}$

4. [NT 6-4.2] Prove that if $f(n)$ is multiplicative, then $\sum_{d|n} \mu(d) f(d) = \prod_{p|n} (1 - f(p))$.

5. *Primitive Roots I.*

   (a) [NT 7-1.1] Find all primitive roots modulo 5, modulo 9, modulo 11, modulo 13, and modulo 15.

   (b) Let a and m be positive, relatively prime integers. Let S be the set of primes dividing $\phi(m)$. Prove that if $a^{\phi(m)/p} \not\equiv 1 \pmod m$ for each $p \in S$, then a is a primitive root of m.

6. *Primitive Roots II.* Let x and y be relatively prime integers.

   (a) Prove that if g is a primitive root modulo xy, then g is a primitive root modulo x and a primitive root modulo y.

   (b) Let h_x be the order of a modulo x and let h_y be the order of a modulo y. In terms of h_x and h_y, find (with proof of correctness) a formula for the order of a modulo xy.

   (c) Use parts (a) and (b) to show that if m is divisible by two distinct odd primes, then there are no primitive roots modulo m.

7. [NT 8-1.4] Modify the proof of Theorem 8–1 to prove that there exist infinitely many primes congruent to $5 \pmod 6$.

8. **[Challenge]** Prove that if n divides $3^n - 1$, then $n = 1$ or n is even.

9. **[Challenge]** The Fibonacci sequence is defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Prove that for each positive integer m, there are infinitely many Fibonacci numbers that are divisible by m.