Name: Solutions

Directions: Show all work. Answers without work generally do not earn points.

1. **[3 parts, 3 points each]** Let $A = \{-9, 2, 4, \{5,6\}, (9,7), \varnothing\}$, let $B = \{n^2 \mid n \in \mathbb{Z}\}$, let $C = \{4, \{6,5\}, (7,9), \{\varnothing\}\}$.

   (a) Determine $|A|$ and $|C|$.

   $|A| = 6$

   $|C| = 4$

   (b) Find $A \cap C$. What is $|A \cap C|$?

   $A \cap C = \{4, \{5,6\}\}$ $\quad |A \cap C| = 2$

   (c) Find $A \cap B$. What is $|A \cap B|$?

   $A \cap B = \{4\}$ $\quad |A \cap B| = 1$

2. **[5 points]** Let $A = \{1\}$. Find $\mathcal{P}(\mathcal{P}(A))$.

   $\mathcal{P}(A) = \{\varnothing, \{1\}\}$

   $\mathcal{P}(\mathcal{P}(A)) = \{\varnothing, \{\varnothing\}, \{\{1\}\}, \{\varnothing, \{1\}\}\}$

3. **[5 points]** Let A be the set of all subsets of $\{1,2,3,4,5\}$ that do not contain two consecutive integers. List the elements of A. What is $|A|$?

$A = \{\varnothing, \{1\}, \{2\}, \{3\}, \{4\}, \{5\},$
$\{1,3\}, \{1,4\}, \{1,5\}, \{2,4\}, \{2,5\}, \{3,5\},$
$\{1,3,5\}\}$

$\boxed{|A| = 13}$

4. **[5 parts, 3 points each]** A town of n people needs to form a committee of k people with a leader. (The leader must be one of the committee members.)

   (a) Suppose we choose the k committee members first and then we choose a leader from the committee. Using this scheme, determine the number of ways to select a committee of size k with a leader.

   $$\boxed{\binom{n}{k} k}$$

   choose committee

   choose leader

   (b) Suppose we choose the leader first and then choose the rest of the committee. Using this scheme, determine the number of ways to select a committee of size k with a leader.

   $$\boxed{n \binom{n-1}{k-1}}$$

   choose leader

   from the remaining $n-1$ people, choose $k-1$ more people to complete the committee

   (c) What conclusion, if any, can you draw from parts (a) and (b)?

   $$\boxed{k\binom{n}{k} = n\binom{n-1}{k-1}}$$

   Both sides count the same thing

   (d) The town decides the size of the committee is no longer important. Using the scheme where the leader is chosen first, count the number of ways to form a committee *of any size* with a leader.

   $$\boxed{n\, 2^{n-1}}$$

   choose leader

   choose a subset of any size for the rest of the committee

   (e) Find a simple formula for the sum $\sum_{k=0}^{n} k\binom{n}{k}$.

   $\sum_{k=0}^{n} k\binom{n}{k}$ adds up # of committees with leader of size k over all possible sizes. So:

   $$\boxed{\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}}$$

5. **[3 parts, 4 points each]** Recall that $\mathbb{N} \times \mathbb{N} = \{(x, y) \mid x \in \mathbb{N} \text{ and } y \in \mathbb{N}\}$. Each of the following parts claims to list the elements of $\mathbb{N} \times \mathbb{N}$ (and therefore prove that $\mathbb{N} \times \mathbb{N}$ is countable). Decide whether or not each list is correct. If incorrect, describe why.

   (a) Begin by listing the pairs where $y = 0$, so that the list begins $(0,0), (1,0), (2,0), \ldots$. Next, list all the pairs where $y = 1$, so that the list continues $(0,1), (1,1), (2,1), \ldots$. Next list all the pairs where $y = 2$, and so on.

   Incorrect. We will never finish with $(0,0), (1,0), (2,0) \ldots$ and move on to pairs (x, y) with $y \geq 1$.

   (b) Begin by listing all the pairs (x, y) where $\max(x, y) = 0$, so that the list begins $(0,0)$. Next, list all the pairs where $\max(x, y) = 1$, so that the list continues $(1,0)$, $(1,1)$, $(0,1)$. Next, list all the pairs where $\max(x, y) = 2$, and so on.

   Correct.

   (c) For each possible value of x, we iterate over all values of y from 0 to x. The list begins $(0,0), (1,0), (1,1), (2,0), (2,1), (2,2), (3,0), \ldots$.

   Incorrect. We will not list pairs (x, y) where $y > x$.

6. **[4 points]** Why did mathematicians switch from Naive Set Theory to Axiomatic Set Theory?

Russell's paradox shows that Naive Set Theory is inconsistent — if we are allowed to have the set $R = \{A : A \notin A\}$, then neither $R \in R$ nor $R \notin R$ is possible.

7. Let $\Sigma = \{0, 1\}$.

   (a) **[3 points]** What is $|\Sigma^3|$?

   $$|\Sigma^3| = \left|\left\{ \underline{x_1}\ \underline{x_2}\ \underline{x_3} : x_i \in \{0,1\} \right\}\right| = 2^3 = \boxed{8}$$

   2 choices for each

   (b) **[3 points]** Write down the set Σ^0 explicitly.

   $$\Sigma^0 = \boxed{\{\lambda\}}$$

   The empty string has length 0.

   (c) **[4 points]** Let A be the set of all strings over Σ of even length and let B be the set of all strings over Σ of odd length. Give a simple English description for the language AB.

   Since an even plus an odd is odd, AB is the set of strings of odd length. That is, $AB = B$.

8. **[2 parts, 4 points each]** Let $\Sigma = \{a, b, c\}$. Let D be the set of all words over Σ in which every a appears before every b. For example, *aacaccacbbb* and *bbcb* are both in D but *bbab* is not in D. Let E be the set of all words over Σ in which every b appears before every a.

   (a) Is it true that $D \cup E = \Sigma^*$? Explain why or why not.

   No. A string with alternating a's and b's, such as aba, is not in $D \cup E$.

   (b) Give a simple, English description for the language $D \cap E$.

   $D \cap E$ is the set of all strings that either have no a's or have no b's.

9. **[3 parts, 4 points each]** Let $\Sigma = \{0, 1\}$ and let A be the language over Σ defined recursively as follows:

1. $\lambda \in A$
2. If $x \in A$, then $x0x \in A$.
3. If $x \in A$, then $x1x \in A$.

(a) List all words in A of length at most 3.

$\boxed{\lambda, 0, 1, 000, 010, 101, 111}$

(b) How many words in A have length 7?

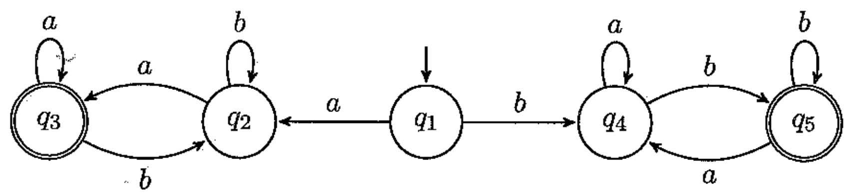
Apply (2) or (3) to one of the 4 strings in A of length 3:

$2 \cdot 4 = \boxed{8}$

(c) Give an example of a word of length 7 that is a palindrome but is not in A.

$\boxed{0110110}$

10. Let $\Sigma = \{a, b\}$ and let M be the following automaton.



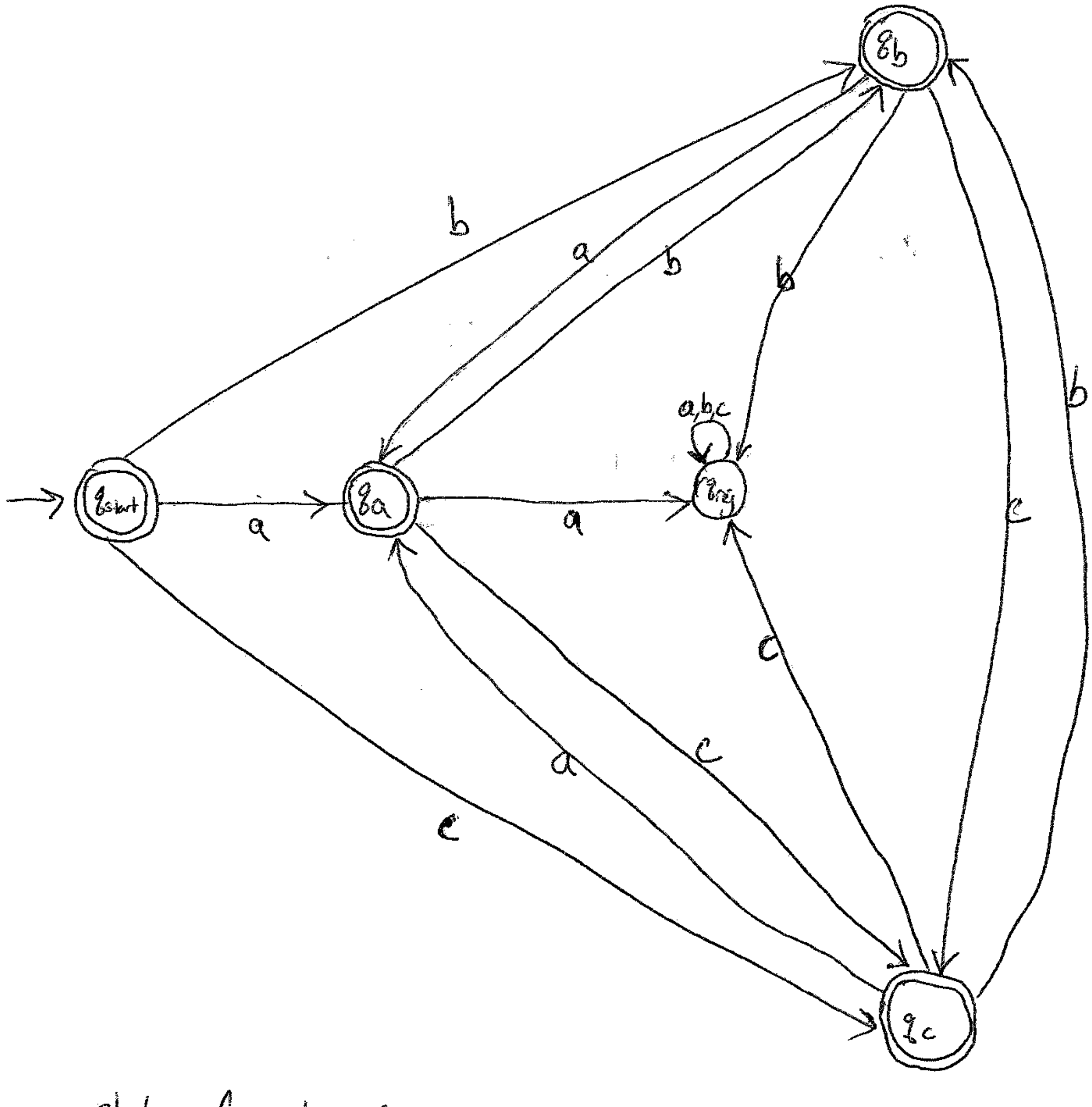
(a) **[4 points]** List the sequence of states that results when M is given $abbab$ as input. Is $abbab \in L(M)$?

States: $q_1, q_2, q_2, q_2, q_3, q_2$ No, $abbab \notin L(M)$

(b) **[6 points]** Give a simple English description of $L(M)$.

L(M) is the set of all strings ~~that~~ of length at least 2 that start and end with the same symbol.

11. **[10 points]** Let $\Sigma = \{a, b, c\}$, and let A be the language of all strings over Σ that do not contain consecutive repeated symbols. For example, $abacbcba \in A$ but $abbac \notin A$. Construct a finite automaton that recognizes A.



Keep states for the last symbol seen.