Name: Solutions

Directions: Show all work. No credit for answers without work.

1. **[2 points]** If a connected planar graph with 83 vertices and 197 edges is drawn in the plane with no edge crossings, how many faces are created? (Include the unbounded, outer face.)

$v - e + f = 2$

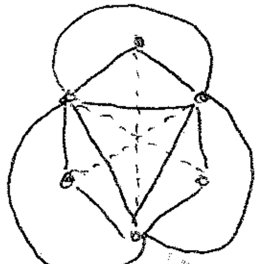
$83 - 197 + f = 2$

$f = 2 + 197 - 83$

$= 199 - 83 = \boxed{116}$

2. **[2 points]** What is the minimum number of edges that must be removed from K_6 (the complete 6-vertex graph) to obtain a planar graph? Argue that your answer is correct.

- K_6 has $\binom{6}{2} = \frac{6\cdot 5}{2} = 15$ edges
- A 6-vertex planar graph can have at most $3(6)-6$ or 12 edges.
- Therefore at least 3 edges must be removed.



K_6 minus three edges (dashed)

Since we can remove 3 edges from K_6 and get a planar graph, the min # of edges to remove is $\boxed{3}$.

3. **[2 points]** Suppose that a 94-vertex planar graph G can be drawn so that the boundary of every face contains at least 6 edges. Prove that G has at most 138 edges.

By adding edges, we may assume G is connected, so $n - e + f = 2$. Also

$$\sum_{\text{Faces } F} \text{length}(F) = 2e$$

$$\sum_{\text{Faces } F} 6 \le 2e$$

$$6f \le 2e$$

$$3f \le e.$$

So $3n - 3e + 3f = 6$ and $3n - 3e + e \ge 6$,

or $3n - 6 \ge 2e$.

Therefore

$$e \le \frac{3n-6}{2}$$

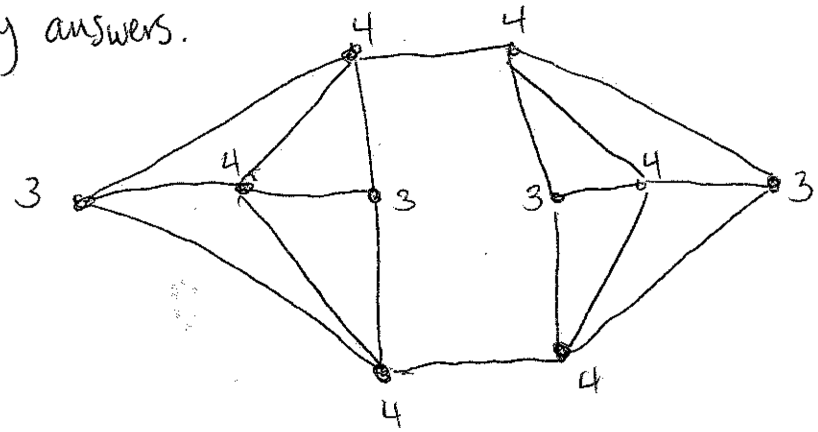
$$\le \frac{3(94)-6}{2}$$

$$\le 3(47) - 3$$

$$\le 3(46) = \boxed{138}.$$

4. **[2 points]** Give a planar drawing of a 10-vertex planar graph in which 6 vertices have degree 4 and the other 4 vertices have degree 3.

Many answers.



5. **[2 points]** Prove that every 10-vertex planar graph in which 6 vertices have degree 4 and the other 4 vertices have degree 3 contains a triangle.

Let G be a planar graph with 6 vertices of degree 4 and 4 vertices of degree 3. Then

$$2e = \sum_{v \in V(G)} d(v) = 6 \cdot 4 + 4 \cdot 3 = 36$$

So G has 18 edges. A 10-vertex graph with more than $2(10) - 4$, or 16, edges must contain a triangle. Since G has more than 16 edges, G contains a triangle.