

Directions: Solve 5 of the following 6 problems. All written work must be your own, using only permitted sources. See the “General Guidelines and Advice” on the homework page for more details.

1. [IGT 1.3.22] Let G be a nonbipartite triangle-free graph with n vertices and minimum degree k . Let l be the minimum length of an odd cycle in G .
 - (a) Let C be a cycle of length l in G . Prove that every vertex not in $V(C)$ contains at most two neighbors in $V(C)$.
 - (b) By counting the edges joining $V(C)$ and $V(G) - V(C)$ in two ways, prove that $n \geq kl/2$ (and thus $l \leq 2n/k$).
 - (c) When k is even, prove that the inequality of part (b) is best possible. (Hint: form a graph having $k/2$ pairwise disjoint l -cycles.)
2. [IGT 1.3.44] Let G be a graph with average degree a . Recall that $a = 2|E(G)|/|V(G)|$.
 - (a) Prove that $G - x$ has average degree at least a if and only if $d(x) \leq a/2$.
 - (b) Give an algorithmic proof that if $a > 0$, then G has a subgraph with minimum degree greater than $a/2$.
 - (c) Show that there is no constant c greater than $1/2$ such that G must have a subgraph with minimum degree greater than ca ; this proves that the bound in part (b) is best possible. Hint: Use $K_{1,n-1}$.
3. [1.3.61] Suppose that G is an n -vertex self-complementary graph and that $n \equiv 1 \pmod{4}$. Prove that G has at least one vertex of degree $(n-1)/2$.
4. [1.3.64] Let $d_1 \leq \dots \leq d_n$ be the vertex degrees of a simple graph G . (Note the non-traditional order.) Prove that G is connected if $d_j \geq j$ when $j \leq n-1-d_n$. (Hint: consider a component that omits a vertex of maximum degree.)
5. [IGT 1.4.25]
 - (a) Prove that every connected graph has an orientation in which the number of vertices with odd outdegree is at most 1. (Hint: consider an orientation with the fewest number of vertices with odd outdegree.)
 - (b) Use part (a) to conclude that every connected graph with an even number of edges has a P_3 -decomposition.
6. [IGT 1.4.29] Suppose that G is a graph and D is an orientation of G that is strongly connected. Prove that if G has an odd cycle, then D has an odd cycle. (Hint: consider each pair $\{v_i, v_{i+1}\}$ in an odd cycle $v_1 \dots v_k$ of G .)