Solutions

1. Solve the IVP $y'' + 4y' + 4y = 0$ with $y(0) = 4$ and $y'(0) = 1$.

Homogeneous.

$r^2 + 4r + 4 = 0$

$(r+2)(r+2) = 0$

$r = -2$ with mult. 2

$y_1 = e^{-2t} \quad y_2 = te^{-2t}$

$y = c_1 e^{-2t} + c_2 t e^{-2t}$.

$y' = -2c_1 e^{-2t} + c_2 e^{-2t} - 2c_2 t e^{-2t}$

$y(0) = 4$: $\quad 4 = c_1 \cdot e^{0} + \cancel{c_2 \cdot 0\, e^{0}}$

$\boxed{4 = c_1}$

$y'(0) = 1$: $\quad 1 = -2c_1 e^{0} + c_2 e^{0} - \cancel{2c_2 0 \cdot e^{0}}$

$1 = -2c_1 + c_2$

$1 = -8 + c_2$

$c_2 = 9$

$$\boxed{y = 4e^{-2t} + 9te^{-2t}}$$

2. Find the general solution to $y^{(3)} - \cancel{6}\,4y^{(2)} + y' + 26y = 0$.

Homogeneous

- $r^3 - 4r^2 + r + 26 = 0$.

- RRT: $\frac{p}{q} = \pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{13}{1}, \pm\frac{26}{1}$

(Since p divides 26 and q divides 1)

- ± 1: Does not work

- $(2)^3 - 4(2)^2 + 2 + 26 \neq 0$.

- $(-2)^3 - 4(-2)^2 - 2 + 26 = 0$ ✓

- So $r - (-2)$ divides $r^3 - 4r^2 + r + 26$.

$\Rightarrow \quad y_1 = e^{-2t}$

$$\begin{array}{r} r^2 - 6r + 13 \\ r+2 \,\big)\, r^3 - 4r^2 + r + 26 \\ -(r^3 + 2r^2) \\ \hline -6r^2 + r \\ -(-6r^2 - 12r) \\ \hline 13r + 26 \\ 13r + 26 \\ \hline 0 \end{array}$$

$(r+2)(r^2 - 6r + 13) = 0$

quad. formula: $r = \dfrac{6 \pm \sqrt{36 - 4 \cdot 13}}{2}$

$r = \dfrac{6 \pm \sqrt{-16}}{2} = 3 \pm 2i$.

Take one from the conjugate pair

$y = e^{(3+2i)t} = e^{3t}(\cos(2t) + i\sin(2t))$.

So $\boxed{y = c_1 e^{-2t} + c_2 e^{3t}\cos(2t) + c_3 e^{3t}\sin(2t)}$

3. Find a particular solution to $y'' + 3y' - 10y = 1 + \cos t$. Hint: first find a family of solutions that containst $\cos^2 t$ and is closed under differentiation.

$1 + \cos t$

$\cancel{0} - \sin t$

$\cancel{-\cos t}$

$Y = A + B\cos t + C\sin t$

$Y' = -B\sin t + C\cos t$

$Y'' = -B\cos t - C\sin t$

$=$

$Y'' + 3Y' - 10Y = 1 + \cos t$

$[-B\cos t - C\sin t] + 3[B\sin t + C\cos t] - 10[A + B\cos t + C\sin t] = 1 + \cos t$

$-10A + (-B + 3C - 10B)\cos t + (-C - 3B - 10C)\sin t = 1 + \cos t$

Set coeffs Equal:

$-10A = 1$; $A = -\frac{1}{10}$

$-11B + 3C = 1$

$[-3B - 11C = 0] \cdot -\frac{11}{3}$

$0B + \frac{130}{3}C = 1$; $C = \frac{3}{130}$

$-11B = 1 - 3C$

$-11B = 1 - \frac{9}{130}$; $B = -\frac{1}{11} \cdot \frac{121}{130} = -\frac{11}{130}$

So $\boxed{Y(t) = -\frac{1}{10} - \frac{11}{130}\cos(t) + \frac{3}{130}\sin(t)}$

4. Find the general solution to $y'' + 3y' - 10y = \cos^2 t$. $1 + \cos t$

(1) Particular soln $Y(t)$ from #3

(2) Solve corresponding homogeneous eqn:

$y'' + 3y' - 10y = 0$

$r^2 + 3r - 10 = 0$

$(r+5)(r-2) = 0$

$r = -5 \quad r = 2$

$y_1 = e^{-5t} \quad y_2 = e^{2t}$

$y = c_1 e^{-5t} + c_2 e^{2t}$

(3) Gen soln is

$\boxed{y(t) = c_1 e^{-5t} + c_2 e^{2t} - \frac{1}{10} - \frac{11}{130}\cos(t) + \frac{3}{130}\sin(t)}$