Solutions. (1)

1. Convert $3u'' - 2u' + 5u = 4$ into a system of first order linear differential equations with constant coefficients.

2. A 2×2 system with real values.

   (a) Find the general solution to

   $$\begin{aligned} x_1' &= -7x_1 + 10x_2 \\ x_2' &= -5x_1 + 8x_2 \end{aligned}.$$

   (b) Draw a phase portrait for the system above.

   (c) Find the solution with initial conditions $x_1(0) = 1$, $x_2(0) = -1$.

3. [7.5.14] Find the general solution to $\mathbf{x}' = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix} \mathbf{x}$.

4. [7.5.10] Complex entries. Find the general solution to $\mathbf{x}' = \begin{bmatrix} 2 & 2+i \\ -1 & -1-i \end{bmatrix} \mathbf{x}$.

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1. Let $x_1 = u$ and $x_2 = u'$. We get

$$x_1' = x_2 \qquad 3x_2' - 2x_2 + 5x_1 = 4$$

or

$$\boxed{\begin{aligned} x_1' &= x_2 \\ x_2' &= \frac{-5}{3}x_1 + \frac{2}{3}x_2 + \frac{4}{3} \end{aligned}}$$

2. (a) Find eigenvalue/eigenvector pairs:

$$\begin{vmatrix} -7-\lambda & 10 \\ -5 & 8-\lambda \end{vmatrix} = 0$$

$$(-7-\lambda)(8-\lambda) - (-5)(10) = 0$$

$$(\lambda - 8)(\lambda + 7) + 50 = 0$$

$$\lambda^2 - \lambda - 6 = 0$$

$$(\lambda - 3)(\lambda + 2) = 0$$

$$\lambda = -2, 3.$$

$\underline{\lambda = -2}$: $\begin{bmatrix} -5 & 10 \\ -5 & 10 \end{bmatrix} \sim \begin{bmatrix} -5 & 10 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$

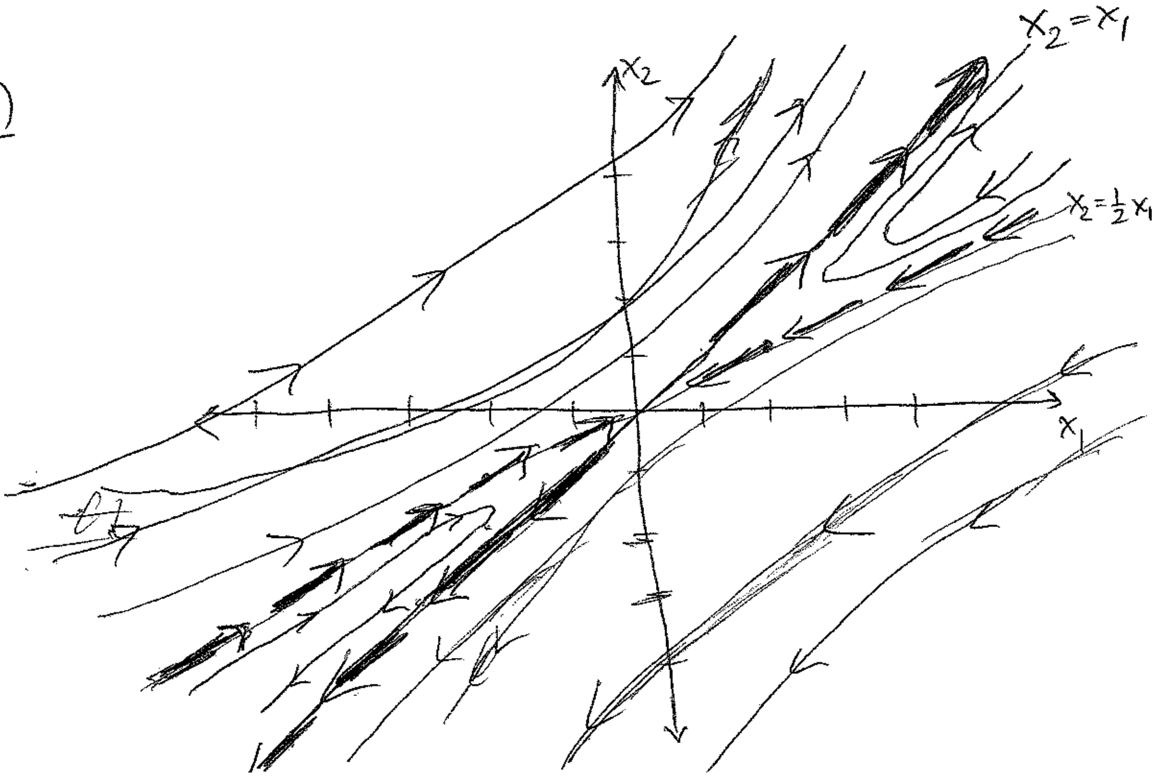
$$\xi = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\underline{\lambda = 3}$: $\begin{bmatrix} -10 & 10 \\ -5 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$$\xi = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$\underline{\text{General}}$ soln: $\boxed{x(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}}$

(b)



(c). Impose $x(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$:

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \cancel{e^{0}}^{\,1} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cancel{e^{0}}^{\,1}$$

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 1 & 1 & -1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 1 & -1 \\ 2 & 1 & 1 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & -1 & 3 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \end{array}\right] \qquad \begin{array}{l} c_1 = 2 \\ c_2 = -3. \end{array}$$

$$\boxed{x(t) = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-2t} - 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}}$$

3. Find Eigenvalue/Eigenvector pairs.

$$\begin{vmatrix} 1-\lambda & -1 & 4 \\ 3 & 2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{vmatrix} = 0$$

$$\left[(1-\lambda)(2-\lambda)(-1-\lambda) + 2 + 12 \right] - \left[8(2-\lambda) - (1-\lambda) - 3(-1-\lambda) \right] = 0$$

$$-(\lambda-1)(\lambda-2)(\lambda+1) + 14 - \left[-4\lambda + 18 \right] = 0$$

$$-(\lambda^3 - 2\lambda^2 - \lambda + 2) + 14 + 4\lambda - 18 = 0$$

$$\lambda^3 - 2\lambda^2 - \lambda + 2 \underline{-14} - 4\lambda \underline{+18} = 0$$

$$\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$$

Rational roots: $\lambda = \frac{p}{q}$ where $p | 6$ and $q | 1$. Try $\lambda = \pm 1, 2, 3, 6.$

$\lambda = 1$: $1 - 2 - 5 + 6 = 0$ ✓.

$$
\begin{array}{r|l}
 & \lambda^2 - \lambda - 6 \\
\hline
\lambda - 1 & \lambda^3 - 2\lambda^2 - 5\lambda + 6 \\
 & \underline{-(\lambda^3 - \lambda^2)} \\
 & \quad -\lambda^2 - 5\lambda \\
 & \quad \underline{-(-\lambda^2 + \lambda)} \\
 & \qquad -6\lambda + 6 \\
 & \qquad \underline{-(-6\lambda + 6)} \\
 & \qquad\qquad 0
\end{array}
$$

$$\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda - 1)(\lambda^2 - \lambda - 6) = 0$$

$$(\lambda - 1)(\lambda - 3)(\lambda + 2) = 0$$

$\lambda = -2, 1, 3.$

$\lambda = -2$: $\begin{bmatrix} 3 & -1 & 4 \\ 3 & 4 & -1 \\ 2 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 3 & -1 & 4 \\ 0 & 5 & -5 \\ 2 & 1 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -2 & 3 \\ 0 & 5 & -5 \\ 2 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 \\ 0 & 5 & -5 \\ 0 & 5 & -5 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$x_1 + x_3 = 0$
$x_2 - x_3 = 0$; Choose $x_3 = 1$. $\xi = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

$\lambda = 1$: $\begin{bmatrix} 0 & -1 & 4 \\ 3 & 1 & -1 \\ 2 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 0 & -1 & 4 \\ 1 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 4 \\ 2 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -4 \\ 0 & 1 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix}$

$x_1 + x_3 = 0$
$x_2 - 4x_3 = 0$; Choose $x_3 = 1$. $\xi = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$

$\lambda = 3$: $\begin{bmatrix} -2 & -1 & 4 \\ 3 & -1 & -1 \\ 2 & 1 & -4 \end{bmatrix} \sim \begin{bmatrix} -2 & -1 & 4 \\ 1 & -2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

$$\rightsquigarrow \begin{bmatrix} 1 & -2 & 3 \\ -2 & -1 & 4 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -2 & 3 \\ 0 & -5 & 10 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 - x_3 = 0$$
$$x_2 - 2x_3 = 0.$$

Try $x_3 = 1$ $\xi = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

So $$\boxed{x(t) = c_1 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} e^{t} + c_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} e^{3t}}$$

4. Find eigenvalue/eigenvector pairs.

$$\begin{vmatrix} 2-\lambda & 2+i \\ -1 & -1-i-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(-1-i-\lambda) - (-1)(2+i) = 0$$

$$(\lambda-2)(\lambda+(1+i)) + (2+i) = 0$$

$$\lambda^2 + (-2+(1+i))\lambda - 2(1+i) + (2+i) = 0$$

$$\lambda^2 + (-1+i)\lambda - i = 0$$

Use quad. formula:

$$\lambda = \frac{1 - i \pm \sqrt{(-1+i)^2 - 4(-i)}}{2}$$

$$= \frac{1 - i \pm \sqrt{1 - 2i + i^2 + 4i}}{2}$$

$$= \frac{1 - i \pm \sqrt{2i}}{2}$$

$$= \frac{1}{2}\left(1 - i \pm \sqrt{2e^{(\pi/2)i}}\right)$$

$$= \frac{1}{2}\left(1 - i \pm \sqrt{2}\, e^{\frac{\pi}{4} i}\right)$$

$\Rightarrow$ Trick: $i = 0 + 1 \cdot i$

$= \cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2})$

$= e^{\pi/2\, i}$

$\Rightarrow \sqrt{e^{\pi/2 i}} = (e^{\frac{\pi}{2} i})^{\frac{1}{2}} = e^{\frac{\pi}{4} i}$

$$= \frac{1}{2}\left(1 - i \pm \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)\right)$$

$$= \frac{1}{2} - \frac{1}{2}i \pm \frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2} + i \cdot \frac{\sqrt{2}}{2}\right)$$

$$= \frac{1}{2} - \frac{1}{2}i \pm \left(\frac{2}{4} + \frac{2}{4}i\right)$$

$\lambda = \left(\frac{1}{2} - \frac{1}{2}i\right) + \left(\frac{2}{4} + \frac{2}{4}i\right)$ or $\lambda = \left(\frac{1}{2} - \frac{1}{2}i\right) - \left(\frac{2}{4} + \frac{2}{4}i\right)$

$\lambda = 1$ or $\lambda = -i$.

Find eigenvectors:

$\underline{\lambda=1}$: $\begin{bmatrix} 1 & 2+i \\ -1 & -2-i \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2+i \\ 0 & 0 \end{bmatrix}$

$x_1 + (2+i)x_2 = 0$; let $x_2 = 1$. $\xi = \begin{bmatrix} -2-i \\ 1 \end{bmatrix}$.

$\underline{\lambda=-i}$: $\begin{bmatrix} 2+i & 2+i \\ -1 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $x_1 + x_2 = 0$ $\xi = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

So $\boxed{x(t) = c_1 \begin{bmatrix} -2-i \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-it}}$.