Name: Solutions

1. **[2 points]** Express $\cos(3t) - \cos(5t)$ as the product of two trigonometric functions.

①
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$-\left[\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta\right]$$
$$\cos(\alpha+\beta) - \cos(\alpha-\beta) = -2\sin\alpha\sin\beta.$$

②
$$\alpha+\beta = 3t$$
$$\alpha-\beta = 5t$$
$$2\alpha = 8t$$
$$\alpha = 4t$$
$$\beta = -t.$$

$$-2\sin(4t)\sin(-t)$$
$$= \boxed{2\sin(4t)\sin(t)}$$

2. **[3 points]** An undamped spring/mass system is modeled by $u'' + 9u = \sin(3t)$ with $u(0) = 0$ and $u'(0) = 0$. Determine $u(t)$. What happens to the system in the limit as t grows?

① $r^2 + 9 = 0$

$r^2 = -9$

$r = \pm 3i$

$u_c = c_1\cos(3t) + c_2\sin(3t)$

② $\sin(3t)$

$3\cos(3t)$

$-9\cancel{\sin(3t)}.$

There is a conflict; multiply by t to get

$9\cdot\left[U(t) = At\sin(3t) + Bt\cos(3t)\right.$

$U' = A\sin(3t) + 3At\cos(3t) + B\cos(3t) - 3Bt\sin(3t)$

$= -3Bt\sin(3t) + 3At\cos(3t) + A\sin(3t) + B\cos(3t)$

$1\cdot\left[U'' = -9At\cos(3t) - 9Bt\cos(3t) - 6B\sin(3t) + 6A\cos(3t)\right]$

$U'' + 9U = 0 + 0 - 6B\sin(3t) + 6A\cos(3t)$

③ $-6B\sin(3t) + 6A\cos(3t) = \sin(3t)$

$\left.\begin{array}{l}-6B = 1\\ 6A = 0\end{array}\right\} \rightarrow A = 0,\ B = -\frac{1}{6}.$

$U(t) = -\frac{1}{6}t\cos(3t).$

④ $u(t) = c_1\cos(3t) + c_2\sin(3t) - \frac{1}{6}t\cos(3t).$

$u(0)=0$: $0 = c_1 + c_2\cdot 0 - \frac{1}{6}\cdot 0\cdot 1$

$\Rightarrow c_1 = 0.$

$u(t) = c_2\sin(3t) - \frac{1}{6}t\cos(3t)$

$u'(t) = 3c_2\cos(3t) - \frac{1}{6}\cos(3t) + \frac{1}{2}t\sin(3t)$

$u'(0)=0$: $0 = 3c_2\cdot 1 - \frac{1}{6}\cdot 1 + \frac{1}{2}\cdot 0$

$3c_2 = \frac{1}{6}$; $c_2 = \frac{1}{18}.$

So:

$$\boxed{u(t) = \frac{1}{18}\sin(3t) - \frac{1}{6}t\cos(3t)}$$

As $t \to \infty$, $|u(t)|$ grows without bound.

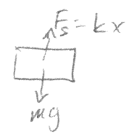
3. A 10 kg mass stretches a spring by 8 cm. The system is contained in a viscous medium which imparts a damping force of 2 N when the mass moves at 10cm/s. A motor imparts an external force of $4\cos(8t)$.

(a) **[3 points]** Find the forced response $U(t)$ with U in m and t in s. Approximate coefficients to 7 decimal places. Units: m, s, kg; force = N.

① $mu'' + \gamma u' + ku = 4\cos(8t)$

- $m = 10\,kg$
- $F_d = \gamma u'$; $2\ \frac{kg\,m}{s^2} = \gamma \cdot \frac{10\,cm}{s} \cdot \frac{m}{100\,cm}$

$\gamma = 20\ \frac{kg}{s}$



$mg = kx$

$10\,kg \cdot 9.8\,m/s^2 = k\ \frac{8\,cm}{100\,cm}\,m$

$k = \frac{9800}{8}\ \frac{kg}{s^2} = 1225\ \frac{kg}{s^2}$

$10u'' + 20u' + 1225u = 4\cos(8t)$

② Since this is a damped system:

$10r^2 + 20r + 1225 = 0$

$u_c(t) = e^{-at}(c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t))$

No conflicts with $U(t) = A\cos(8t) + B\sin(8t)$.

③ $[U(t) = A\cos(8t) + B\sin(8t)] \cdot 1225$

$[U' = 8B\cos(8t) - 8A\sin(8t)] \cdot 20$

$[U'' = -64A\cos(8t) - 64B\sin(8t)] \cdot 10$

$4\cos(8t) = (1225A + 160B - 640A)\cos(8t) + (1225B - 160A - 640B)\sin(8t)$

$585A + 160B = 4$

$[-160A + 585B = 0] \cdot \frac{585}{160}$

$0A + \left(160 + \frac{(585)^2}{160}\right)B = 4$

$B = \frac{(4)(160)}{(160)^2 + (585)^2} \approx 0.0017400$

$A = \frac{585}{160}B \approx 0.0063617$

$$\boxed{U(t) \approx (0.0017400)\cos(8t) + (0.0063617)\sin(8t)}$$

(b) **[1 point]** Express the forced response $U(t)$ in the form $R\cos(\omega t - \delta)$. Approximate R to 5 decimal places and δ to 3.

$R = \sqrt{A^2 + B^2} \approx 0.00660$

④ $\delta = \tan^{-1}\left(\frac{B}{A}\right) \approx 0.267$

$$\boxed{U(t) \approx (0.00660)\cos(8t - 0.267)}$$

(c) **[1 point]** Compare the amplitude of the forced response to the displacement when a constant force of 4 N is applied. Which is larger?

Amplitude of forced response $= \boxed{R \approx 0.00660\ m}$

Const force displacement: $\frac{F}{k} = \frac{4\ \frac{kg\,m}{s^2}}{1225\ \frac{kg}{s^2}} \approx \boxed{0.00327\ m}$

The forced response has a larger magnitude (This is an example of resonance.)