Name: Solutions

1. **[2.5 points]** Solve the initial value problem $y'' + 8y' - 9y = 0$ with $y(0) = 1$ and $y'(0) = 2$. Express your answer using real numbers only.

$r^2 + 8r - 9 = 0$

$(r+9)(r-1) = 0$

$r_1 = -9 \quad r_2 = 1$

$y_1 = e^{-9t} \quad y_2 = e^{t}$

$y = c_1 e^{-9t} + c_2 e^{t}$

$y' = -9c_1 e^{-9t} + c_2 e^{t}$

$y(0)=1:\ -(1 = c_1 \cdot 1 + c_2 \cdot 1)$

$y'(0)=2:\ 2 = -9c_1 \cdot 1 + c_2$

$1 = -10c_1 \qquad c_1 = -\frac{1}{10}$

$1 = -\frac{1}{10} + c_2 \qquad c_2 = \frac{11}{10}$

$y = -\frac{1}{10} e^{-9t} + \frac{11}{10} e^{t}$

2. **[2.5 points]** Find the general solution to $y'' + 2y' + 2y = 0$. Express your answer using real numbers only.

$r^2 + 2r + 2 = 0$

$r = \frac{-2 \pm \sqrt{4 - 4 \cdot 2}}{2}$

$r = \frac{-2 \pm \sqrt{-4}}{2}$

$r = \frac{-2 \pm 2i}{2}$

$r = -1 \pm i$

$y_1 = e^{(-1+i)t} = e^{-t} \cdot e^{ti}$

$y_1 = e^{-t}(\cos(t) + i\sin(t))$

Extract Real & Imaginary parts:

$y_3 = e^{-t}\cos(t) \qquad y_4 = e^{-t}\sin(t)$

Combine:

$\boxed{y = c_1 e^{-t}\cos(t) + c_2 e^{-t}\sin(t)}$

$y = c_1 e^{-t}\cos(t) + c_2 e^{-t}\sin(t)$

3. **[1 point]** Short answer. Suppose that y_1 and y_2 are solutions to $y'' + p(t)y' + q(t) = 0$ on an open interval I. What useful information does the Wronskian of y_1 and y_2 provide?

Ans 1: If $W(y_1, y_2) \neq 0$, then all solns are of the form $y = c_1 y_1 + c_2 y_2$.

OR

Ans 2: If $W(y_1, y_2) \neq 0$, then y_1 and y_2 form a fundamental set of solns.

4. **[2 points]** Compute the Wronskian of $\sin t$ and $t \sin t$.

$$W = \begin{vmatrix} \sin t & t\sin t \\ \cos t & \sin t + t\cos t \end{vmatrix} = \sin t(\sin t + \cancel{t\cos t}) - \cancel{t \sin t \cos t}$$

$$= \boxed{\sin^2 t}.$$

5. **[2 points]** Solve the following differential equation: $x\frac{dy}{dx} + xy = 1 - y$.

Linear First order.

$$x\frac{dy}{dx} + (1+x)y = 1$$

$$\frac{dy}{dx} + \left(\frac{1}{x} + 1\right)y = \frac{1}{x}$$

$$\mu = e^{\int \frac{1}{x} + 1\,dx} = e^{\ln x + x} = xe^x$$

$$xe^x\frac{dy}{dx} + (e^x + xe^x)y = e^x$$

$$\frac{d}{dx}\left[xe^x y\right] = e^x$$

$$xe^x y = \int e^x\,dx$$

$$xe^x y = e^x + C$$

$$\boxed{y = \frac{1}{x} + \frac{C}{x}e^{-x}}$$

or

$$\boxed{y = \frac{1}{x}\left(1 + Ce^{-x}\right)}.$$

or

$$\boxed{y = \frac{1}{x} + \frac{C}{xe^{x}}}$$