1. [EC 12.4.8] Find the mass and center of mass of the lamina occupysing the region D with density ρ where D is bounded by $y = \sqrt{x}$, $y = 0$, and $x = 1$, and $\rho(x, y) = x$.

2. [EC 12.5.4] Evaluate $\int_0^1 \int_x^{2x} \int_0^y 2xyz\, dz\, dy\, dx$.

3. [EC 12.5.12] Evaluate $\iiint_E xz\, dV$ where E is the solid tetrahedron with vertices $(0, 0, 0)$, $(0, 1, 0)$, $(1, 1, 0)$, and $(0, 1, 1)$.

4. [EC 12.6.4(a)] Change from rectangular to cylindrical coordinates: $(3, 3, -2)$.

5. [EC 12.6.18] Evaluate $\iiint_E (x^3 + xy^2)\, dV$, where E is the solid in the first octant (x, y, z are all positive) that lies beneath the paraboloid $z = 1 - x^2 - y^2$.

6. [EC 12.6.28] Evaluate by changing to cylindrical coordinates:

$$\int_{-3}^{3} \int_0^{\sqrt{9-x^2}} \int_0^{9-x^2-y^2} \sqrt{x^2 + y^2}\, dz\, dy\, dx$$

7. [EC 12.7.2a] Change $(5, \pi, \pi/2)$ from spherical coordinates to rectangular coordinates.

8. [EC 12.7.4a] Change $(0, \sqrt{3}, 1)$ from rectangular coordinates to spherical coordinates.

9. [EC 12.7.10] Write the equation in spherical coordinates,

   (a) $x^2 + y^2 + z^2 = 2$

   (b) $z = x^2 - y^2$

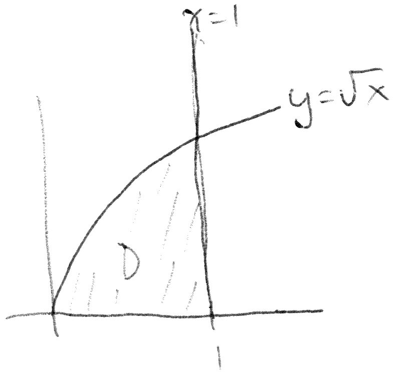
10. [EC 12.7.22] Evaluate $\iiint_H (x^2 + y^2)\, dV$, where H is the hemispherical region that lies above the xy-plane and below the sphere $x^2 + y^2 + z^2 = 1$.

11. [EC 12.7.26] Find the volume of the solid that lies within the sphere $x^2 + y^2 + z^2 = 4$, above the xy-plane, and below the cone $z = \sqrt{x^2 + y^2}$.

Solutions

WS 7 Solns

12



$$\text{mass} = m = \int_0^1 \int_0^{\sqrt{x}} \rho(x,y)\,dy\,dx$$

$$= \int_0^1 \int_0^{\sqrt{x}} x\,dy\,dx$$

$$= \int_0^1 xy\Big|_{y=0}^{y=\sqrt{x}} dx$$

$$= \int_0^1 x(\sqrt{x} - 0)\,dx$$

$$= \int_0^1 x^{3/2}\,dx = \frac{2}{5}x^{5/2}\Big|_{x=0}^{x=1} = \frac{2}{5} - 0 = \boxed{\frac{2}{5}}.$$

$\circ \; M_y = \iint_D x \rho \, dA = \iint_D x \cdot x \, dA$

$= \int_0^1 \int_0^{\sqrt{x}} x^2 \, dy \, dx$

$= \int_0^1 x^2 y \Big|_{y=0}^{y=\sqrt{x}} dx$

$= \int_0^1 x^2 \cdot x^{\frac{1}{2}} dx$

$= \int_0^1 x^{5/2} dx$

$= \frac{2}{7} x^{7/2} \Big|_{x=0}^{x=1} = \frac{2}{7}$

$\circ \; \bar{x} = \frac{M_y}{M} = \frac{\frac{2}{7}}{\frac{2}{5}} = \frac{5}{7}$

$\circ \; M_x = \iint_D y \rho \, dA = \iint xy \, dA$

$= \int_0^1 \int_0^{\sqrt{x}} xy \, dy \, dx$

$= \int_0^1 \frac{x}{2} y^2 \Big|_{y=0}^{y=\sqrt{x}} dx$

$= \int_0^1 \frac{x}{2}(x - 0) \, dx$

$= \frac{1}{2} \int_0^1 x^2 \, dx = \frac{1}{2} \cdot \frac{x^3}{3} \Big|_{x=0}^{1} = \frac{1}{2}\left(\frac{1}{3} - 0\right) = \frac{1}{6}$

$$\bar{y} = \frac{M_x}{m} = \frac{\frac{1}{6}}{\frac{2}{5}} = \frac{5}{12}$$

So center of mass is $\boxed{\left(\frac{5}{7}, \frac{5}{12}\right)}$.

2. $$\int_0^1 \int_x^{2x} \left[\int_0^y 2xyz \, dz\right] dy\, dx$$

$$= \int_0^1 \int_x^{2x} \cancel{xzyz} \; xyz^2 \Big|_{z=0}^{z=y} dy\, dx$$

$$= \int_0^1 \int_x^{2x} xy^3 \, dy\, dx$$

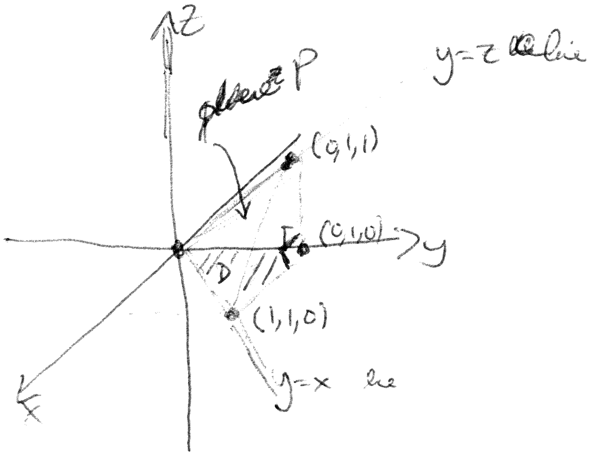
$$= \int_0^1 \frac{x}{4} y^4 \Big|_{y=x}^{y=2x} dx$$

$$= \int_0^1 \frac{x}{4}\left((2x)^4 - x^4\right) dx$$

$$= \int_0^1 \frac{x}{4}\left(16x^4 - x^4\right) dx$$

$$= \int_0^1 \frac{15}{4} x^5 \, dx = \frac{15}{4 \cdot 6} \cdot x^6 \Big|_{x=0}^{1} = \boxed{\frac{15}{24}} = \boxed{\frac{5}{8}}$$

3.



- D is triangle in $z=0$ plane
- For (x,y) in D, z goes from 0 to the plane containing $(0,0,0)$, $(0,1,1)$, $(1,1,0)$
- $\vec{n} = \langle 0,1,1\rangle \times \langle 1,1,0\rangle$

$$= \begin{vmatrix} i & j & k \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -1i + j - k = \langle -1,1,-1\rangle$$

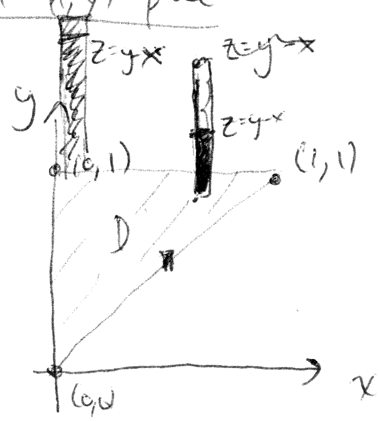
- ~~$D = \{(x,y) \text{ in } x \geq 0,\ y \geq x,\ y \leq 1\}$~~
- $E = \{(x,$
- So, this plane P is:

$$\langle -1,1,-1\rangle \cdot \langle x,y,z\rangle = 0$$

$$-x + y - z = 0.$$

$$z = y - x$$

D in (x,y)-plane:



- $$\iiint_E xz\,dV = \iint_D \left[\int_0^{y-x} xz\,dz\right] dA$$

$$= \iint_D \frac{x}{2} z^2 \Big|_{z=0}^{z=y-x} dA = \iint_D \frac{x}{2}\left((y-x)^2 - 0\right) dA$$

x is a const here

$$\iint_D \frac{x}{2}(y-x)^2\,dA = \int_0^1 \boxed{\int_x^1 \frac{x}{2}(y-x)^2\,dy}\,dx$$

$$= \int_0^1 \frac{x}{2}\cdot\frac{1}{3}(y-x)^3\Big|_{y=x}^{y=1}\,dx$$

$$= \int_0^1 \frac{x}{6}\left((1-x)^3 - 0\right)dx$$

$$= \frac{1}{6}\int_0^1 \frac{x}{6}(1-x)^3\,dx$$

$$u = 1-x\ ,\quad du = -dx$$

$$= \frac{1}{6}\int_1^0 (1-u)\,u^3\,(-du)$$

$$= \frac{1}{6}\int_0^1 u^3 - u^4\ du$$

$$= \frac{1}{6}\left[\frac{u^4}{4} - \frac{u^5}{5}\right]\Big|_{u=0}^{u=1}$$

$$= \frac{1}{6}\left[\left(\frac{1}{4}-\frac{1}{5}\right) - 0\right]$$

$$= \frac{1}{6}\cdot\frac{1}{20} = \boxed{\frac{1}{120}}$$

ex

4. $(x, y, z) = (3, 3, -2)$

$r^2 = x^2 + y^2 = 9 + 9 = 18$

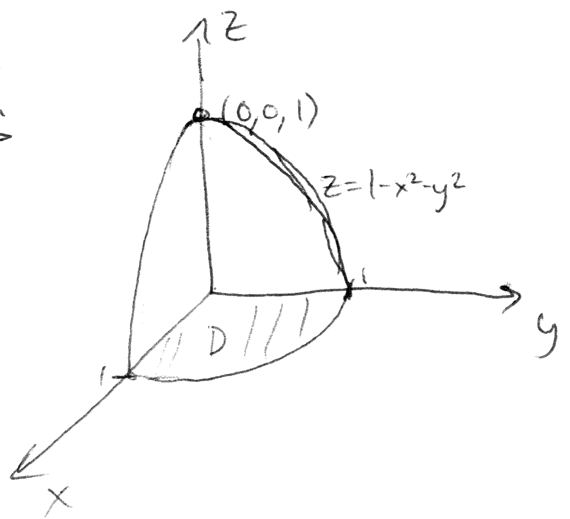
$r = \sqrt{18} = 3\sqrt{2}$

$\tan\theta = \frac{y}{x}$, $\tan\theta = 1 \Rightarrow \theta = \pi/4$

So $(3, 3, -2)$ becomes (r, θ, z)

or $\boxed{(3\sqrt{2}, \pi/4, -2)}$.

5.



• Let $D = \{(x,y) : xy \geq 0, x^2+y^2 \leq 1\}$

$= \{(r,\theta) : 0 \leq r \leq 1, 0 \leq \theta \leq \frac{\pi}{2}\}$

$$\iiint_E (x^3 + xy^2)\,dV = \iint_D \left[\int_0^{1-x^2-y^2} (x^3 + xy^2)\,dz\right] dA$$

$$= \iint_D \left[(x^3 + xy^2) z \right]_{z=0}^{z=1-x^2-y^2} dA$$

$$= \iint_D (x^3 + xy^2)(1 - x^2 - y^2)\, dA$$

$$= \iint_D x(x^2 + y^2)(1 - (x^2+y^2))\, dA$$

$$= \int_0^{\pi/2} \int_0^1 x(x^2+y^2)(1 - (x^2+y^2))\, r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \int_0^1 (r\cos\theta)(r^2)(1 - r^2)\, r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \int_0^1 r^4(1-r^2) \cos\theta \, dr\,d\theta$$

$$= \left(\int_0^{\pi/2} \cos\theta\, d\theta\right)\left(\int_0^1 r^4(1-r^2)\, dr\right)$$

$$= \left[+\sin\theta \right]_{\theta=0}^{\theta=\pi/2} \cdot \left[\frac{r^5}{5} - \frac{r^7}{7} \right]_{r=0}^{r=1}$$

$$= (1 - 0) \cdot \left(\left(\frac{1}{5} - \frac{1}{7}\right) - 0 \right)$$

$$= 1 \cdot \frac{2}{35} = \boxed{\frac{2}{35}}$$

6. $$\int_{-3}^{3}\left[\int_{0}^{\sqrt{9-x^2}}\left[\int_{0}^{9-x^2-y^2} \cdots \, dz\right]dy\right]dx$$

- x ranges from -3 to 3

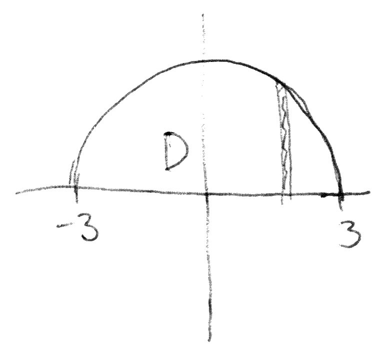
- y ranges from 0 to a max of $\sqrt{9-x^2}$.

$$y = \sqrt{9-x^2}$$

$$y^2 = 9-x^2$$

$$x^2+y^2=9$$

circle of radius 3.



$D = \{(x,y): y \geq 0 \text{ and } x^2+y^2 \leq 9\}$

- z ranges from 0 to a max of $9-x^2-y^2$.

$$z = 9-x^2-y^2$$

- So, with $D =$ semicircle centered at $(0,0)$ with radius 3 and $E = \{(x,y,z): (x,y) \text{ in } D \text{ and } 0 \leq z \leq 9-x^2-y^2\}$,

we want $$\iiint_E \sqrt{x^2+y^2}\, dV.$$

$$\iiint_E \sqrt{x^2+y^2}\, dV = \iint_D \left[\int_0^{9-x^2-y^2} \sqrt{x^2+y^2}\, dz\right] dA$$

$$= \iint_D \left[\left(\sqrt{x^2+y^2}\right) z\right]_{z=0}^{z=9-x^2-y^2} dA$$

$$= \iint_D \sqrt{x^2+y^2} \cdot (9-x^2-y^2)\, dA$$

$$= \int_0^{\pi} \int_0^{3} \sqrt{x^2+y^2}\,(9-(x^2+y^2))\, r\, dr\, d\theta$$

$$= \int_0^{\pi} \int_0^{3} r \cdot (9-r^2) \cdot r\, dr\, d\theta$$

$$= \left(\int_0^{\pi} 1\, d\theta\right)\left(\int_0^3 r^2(9-r^2)\, dr\right)$$

$$= \pi \cdot \left(\int_0^3 9r^2 - r^4\, dr\right)$$

$$= \pi \left(\left(3r^3 - \frac{r^5}{5}\right)_{r=0}^{r=3}\right)$$

$$= \pi \left(\left(81 - \frac{243}{5}\right) - 0\right)$$

$$= \pi \left(\frac{405}{5} - \frac{243}{5}\right) = \pi \frac{162}{5} = \cancel{\boxed{\frac{324}{5}\pi}}\ \boxed{\frac{162}{5}\pi}$$

7. If $(\rho, \theta, \phi) = (5, \pi, \pi/2)$, then

$$z = \rho\cos\phi = 5\cos\tfrac{\pi}{2} = 5\cdot 0 = 0$$

$$y = \rho\sin\phi\sin\theta = 5\sin\tfrac{\pi}{2}\sin(\pi) = 5\cdot 1\cdot 0 = 0$$

$$x = \rho\sin\phi\cos\theta = 5\sin\tfrac{\pi}{2}\cos(\pi) = 5\cdot 1\cdot(-1) = -5$$

So, this point has rectangular coords $\boxed{(-5,0,0)}$.

8. If $(x, y, z) = (0, \sqrt{3}, 1)$, then

- $\rho^2 = x^2+y^2+z^2 = 0^2+3+1^2 = 4$, so $\rho = 2$.

- $z = \rho\cos\phi$, so $1 = 2\cos\phi$

  $\cos\phi = \frac{1}{2}$, so $\phi = \frac{\pi}{3}$

- $x = \rho\sin\phi\cos\theta$, so $0 = 2\left(\sin\frac{\pi}{3}\right)(\cos\theta)$

  $0 = 2\cdot\frac{\sqrt{3}}{2}\cdot\cos\theta$,

  $\cos\theta = 0$, so $\theta = \pi/2$.

This point has spherical coordinates $\boxed{(2, \pi/2, \pi/3)}$.

9. a $x^2+y^2+z^2=2$ Becomes $\rho^2=2$, or $\boxed{\rho=\sqrt{2}}$, the sphere of radius $\sqrt{2}$.

b. $z = x^2-y^2$ becomes

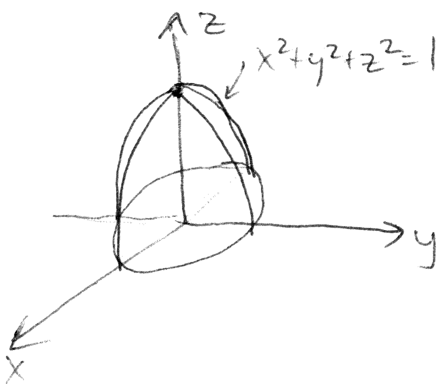
$$\rho\cos\phi = (\rho\sin\phi\cos\theta)^2 - (\rho\sin\phi\sin\theta)^2$$

$$\rho\cos\phi = \rho^2\sin^2\phi(\cos^2\theta - \sin^2\theta)$$

$$\cos\phi = \rho\sin^2\phi\cos(2\theta)$$

$$\boxed{\rho\tan\phi\sin\phi\cos(2\theta) = 1}$$

10.



ensures $z\geq 0$

$$H = \{(\rho,\theta,\phi) : 0\leq\phi\leq\pi/2,\ 0\leq\theta\leq 2\pi,\ 0\leq\rho\leq 1\}$$

$$\iiint_H (x^2+y^2)\,dV = \int_0^{\pi/2}\int_0^{2\pi}\int_0^1 (x^2+y^2)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

$$= \int_0^{\pi/2}\int_0^{2\pi}\int_0^1 \Big(\overbrace{\rho^2 \sin^2\phi}^{r^2}\Big)\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \int_0^{\pi/2}\int_0^{2\pi}\int_0^1 \rho^4 \sin^3\phi \, d\rho \, d\theta \, d\phi$$

Can integrate by parts, or trig substitution

$$= \left(\int_0^1 \rho^4 \, d\rho\right)\left(\int_0^{2\pi} 1 \, d\theta\right)\boxed{\left(\int_0^{\pi/2} \sin^3\phi \, d\phi\right)}$$

$$= \left(\frac{1}{5}\rho^5\right)\Big|_{\rho=0}^{\rho=1} \cdot (2\pi) \cdot \left(\int_0^{\pi/2} \sin\phi \,(1-\cos^2\phi)\, d\phi\right)$$

$$= \frac{2}{5}\pi \cdot \left(\int_0^{\pi/2} \sin\phi \, d\phi - \int_0^{\pi/2} \cos^2\phi \sin\phi \, d\phi\right)$$

$u = \cos\phi$, $du = -\sin\phi \, d\phi$

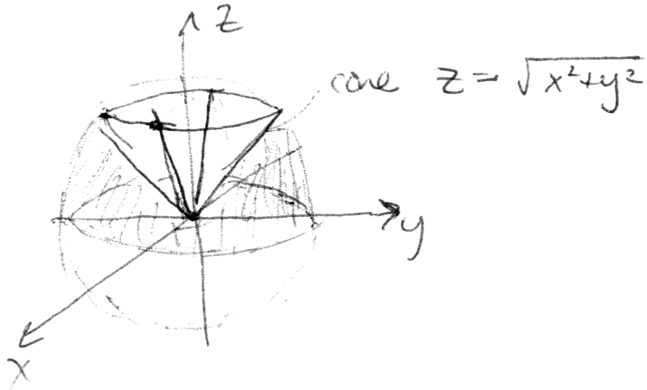
$$= \frac{2}{5}\pi \cdot \left((-\cos\theta)\Big|_{\theta=0}^{\theta=\pi/2} - \int_1^0 u^2 (-du) \right)$$

$$= \frac{2}{5}\pi \cdot \left(\left(-\cos\frac{\pi}{2}\right) - (-\cos 0) - \int_0^1 u^2 \, du \right)$$

$$= \frac{2}{5}\pi \left(0 + 1 - \frac{u^3}{3}\Big|_{u=0}^{u=1} \right)$$

$$= \frac{2}{5}\pi \cdot \left(1 - \left(\frac{1}{3} - 0\right)\right) = \frac{2}{5} \cdot \pi \cdot \frac{2}{3} = \boxed{\frac{4}{15}\pi}$$

11.



• $E = \cancel{\{(\rho, \theta, \phi)}:$

• Restrict $\pi/4 \le \phi \le \pi/2$

below cone   above xy-plane

• No Restriction on θ; $0 \le \theta \le 2\pi$

• $\rho \le 2$

• $E = \{(\rho, \theta, \phi) : \pi/4 \le \phi \le \pi/2,\ 0 \le \theta \le 2\pi,\ 0 \le \rho \le 2\}$

• Volume $= \iiint_E 1 \, dV$

$$= \int_0^2 \int_0^{2\pi} \int_{\pi/4}^{\pi/2} 1 \cdot \rho^2 \sin\phi \, d\phi \, d\theta \, d\rho$$

$$= \left(\int_0^2 \rho^2 \, d\rho\right)\left(\int_0^{2\pi} 1 \, d\theta\right)\left(\int_{\pi/4}^{\pi/2} \sin\phi \, d\phi\right)$$

$$= \left(\frac{\rho^3}{3}\Big|_{\rho=0}^{\rho=2}\right) \cdot 2\pi \cdot \left(-\cos\phi\right)_{\phi=\pi/4}^{\phi=\pi/2}$$

$$= \frac{8}{3} \cdot 2\pi \cdot \left(\left(-\cos\frac{\pi}{2}\right) - \left(-\cos\frac{\pi}{4}\right) \right)$$

$$= \frac{16}{3}\pi \cdot \left(0 + \frac{\sqrt{2}}{2} \right) = \frac{16\sqrt{2}}{6}\pi = \boxed{\frac{8\sqrt{2}}{3}\pi}$$