Name: Solutions

Directions: Show all work. No credit for answers without work.

1. [**6 points**] Let V be the vector space of real-valued functions on the whole real line, over the field of real numbers. Is $\{x^2, x^2 + 3x, 2x\}$ linearly independent in V? Justify your answer.

$\boxed{\text{Linearly Dependent.}}$ We can find a non-trivial linear combination that gives the zero vector:

$$(-1)(x^2) + (1)(x^2+3x) + (-\tfrac{3}{2})(2x) = 0$$

2. [**3 parts, 2 points each**] The following questions are about the vector space $\mathbb{R}^4$.

   (a) Is there a vector $\vec{a}$ in $\mathbb{R}^4$ that is contained in the span of every set of vectors? If so, write down $\vec{a}$ explicitly. If not, explain why not.

   Yes. $\vec{a} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

   (b) Let S be a set of vectors in $\mathbb{R}^4$. Is it true that if $\vec{a} \in S$, then $\vec{a}$ is in the span of S? Explain.

   Yes. We obtain $\vec{a}$ as a linear combination by setting the coefficient of $\vec{a}$ to 1 and all other coefficients to zero.

   (c) Give an example of a basis of $\mathbb{R}^4$.

   The standard basis: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

3. **[6 parts, 2 points each]** Let

$$A = \begin{bmatrix} 3 & 1+i & 2-i \\ 4 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} -2 & 0 \\ 1 & i \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad D = \begin{bmatrix} 3+2i & -7 \\ 0 & 1 \\ 2i & 0 \end{bmatrix}$$

be matrices over the field of complex numbers $\mathbb{C}$. For each of the following, write the specified matrix explicitly if possible, or write "undefined" otherwise.

(a) $A + B$

Undefined

(b) iD

$$\begin{bmatrix} -2+3i & -7i \\ 0 & i \\ -2 & 0 \end{bmatrix}$$

(c) $\overline{A}$ ← complex conjugate

$$\begin{bmatrix} 3 & 1-i & 2+i \\ 4 & 0 & 1 \end{bmatrix}$$

(d) C^T ← transpose

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

(e) AA

Undefined

(f) BB

$$\begin{bmatrix} -2 & 0 \\ 1 & i \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 1 & i \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ -2+i & -1 \end{bmatrix}$$

4. **[12 points]** Using matrices and Gauss-Jordan elimination, find all solutions to the following system of linear equations.

$$\begin{array}{rcrcrcrcrcl}
x_1 & + & 2x_2 & - & 2x_3 & + & 6x_4 & & & = & 0 \\
 & & 2x_2 & + & 4x_3 & - & 2x_4 & \cancel{+} & \cancel{x_5} & = & 0 \\
2x_1 & & & - & 12x_3 & + & 16x_4 & \cancel{+} & \cancel{x_5} & = & 0 \\
-2x_1 & + & 3x_2 & + & 18x_3 & - & 19x_4 & \cancel{+} & \cancel{x_5} & = & 0
\end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -2 & 6 & 0 \\ 0 & 2 & 4 & -2 & 0 \\ 2 & 0 & -12 & 16 & 0 \\ -2 & 3 & 18 & -19 & 0 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 2 & -2 & 6 & 0 \\ 0 & 2 & 4 & -2 & 0 \\ 0 & -4 & -8 & 4 & 0 \\ 0 & 7 & 14 & -7 & 0 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & -6 & 8 & 0 \\ 0 & 2 & 4 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 7 & 14 & -7 & 0 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & -6 & 8 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & -6 & 8 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ & & \uparrow & \uparrow \\ & & \text{free} & \text{free} \end{array}$$

$$\boxed{\begin{array}{l} \text{So, the solutions are} \\ x_1 = 6c - 8d \\ x_2 = -2c + d \\ x_3 = c \\ x_4 = d \\ \text{where } c, d \in \mathbb{R}. \end{array}}$$

5. **[6 points]** Find a matrix in Reduced Row Echelon Form that is row-equivalent to the matrix A below.

$$A = \begin{bmatrix} 3 & 15 & -2 & 11 & 2 \\ 2 & 10 & -3 & 9 & -2 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 5 & 1 & 2 & 4 \\ 2 & 10 & -3 & 9 & -2 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 5 & 1 & 2 & 4 \\ 0 & 0 & -5 & 5 & -10 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 5 & 1 & 2 & 4 \\ 0 & 0 & 1 & -1 & 2 \end{bmatrix}$$

$$\rightsquigarrow \boxed{\begin{bmatrix} 1 & 5 & 0 & 3 & 2 \\ 0 & 0 & 1 & -1 & 2 \end{bmatrix}}$$

6. **[6 points]** Consider the vector space $\mathbb{R}^3$ and let $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$. Which vectors $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ are in span(S)? Give a simple condition on b_1, b_2, and b_3 which answers this question.

Solve $\alpha_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \Longleftrightarrow \left[\begin{array}{cc|c} 1 & 4 & b_1 \\ 2 & 5 & b_2 \\ 3 & 6 & b_3 \end{array}\right]$

$$\rightsquigarrow \left[\begin{array}{cc|c} 1 & 4 & b_1 \\ 0 & -3 & b_2 - 2b_1 \\ 0 & -6 & b_3 - 3b_1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 4 & b_1 \\ 0 & 3 & 2b_1 - b_2 \\ 0 & -6 & b_3 - 3b_1 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cc|c} 1 & 4 & b_1 \\ 0 & 3 & 2b_1 - b_2 \\ 0 & 0 & b_3 - 3b_1 + 2(2b_1 - b_2) \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 4 & b_1 \\ 0 & 1 & \frac{2}{3}b_1 - \frac{1}{3}b_2 \\ 0 & 0 & b_1 - 2b_2 + b_3 \end{array}\right]$$

How to set α_1, α_2 $\Longleftrightarrow$

$$\begin{cases} \alpha_1 + 4\alpha_2 = b_1 \\ \alpha_2 = \frac{2}{3}b_1 - \frac{1}{3}b_2 \\ 0 = b_1 - 2b_2 + b_3 \end{cases}$$

constraint on $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ $\rightarrow$ $0 = b_1 - 2b_2 + b_3$

So $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ is in the span of S if and only if $\boxed{b_1 - 2b_2 + b_3 = 0}$

In other words, $b_2 = \frac{b_1 + b_3}{2}$, so b_2 must be the average of b_1 and b_3