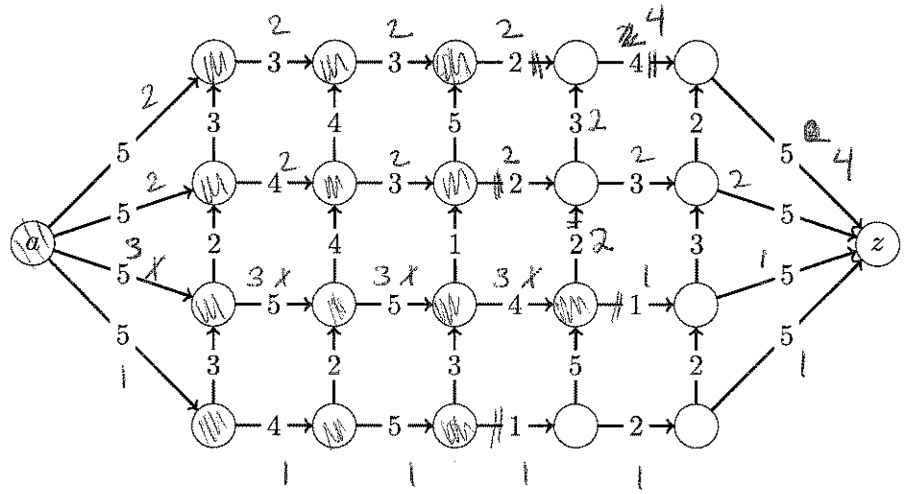
Name: Solutions

Directions: Show all work. Answers without work generally do not earn points. This test has 60 points but is scored out of 50 (higher scores capped at 50).

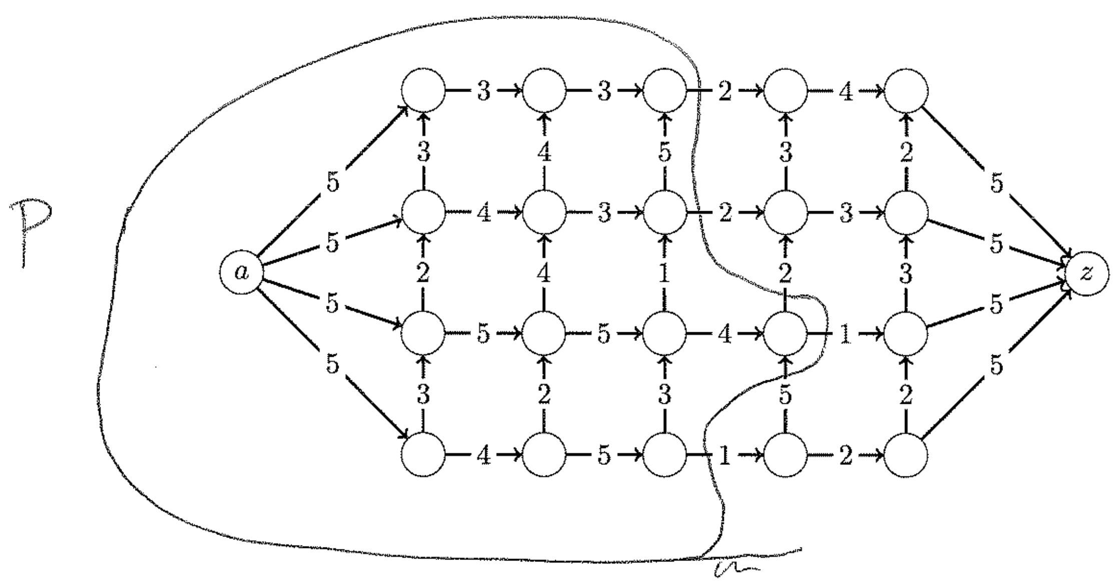
1. [**2 parts, 5 points each**] Max flow/Min Cut.

   (a) Find a flow of maximum value in the following network.



Flow Value 8

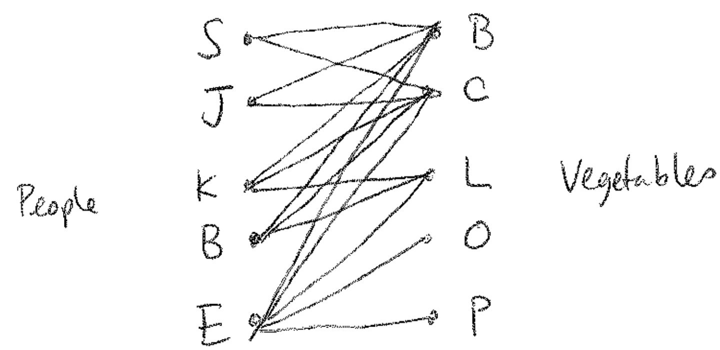
   (b) Find a cut of minimum capacity in the same network (repeated below).



Cut Capacity 8

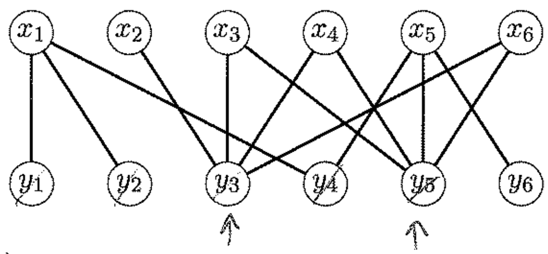
2. [**4 points**] A supermarket stocks 5 different vegetables (broccoli, carrots, lettuce, onions, and peas). At the same time, 5 customers (Beth, Eric, Jerry, Kim, and Sam) enter the store in search of vegetables. Unfortunately, due to limited supplies, it is not possible for more than 1 person to purchase any given vegetable. Sam and Jerry each like 2 vegetables. Kim and Beth each like 3 vegetables. Eric likes all 5 vegetables. Is it guaranteed that everyone can purchase a vegetable they like? Either show that this is the case or find a counterexample.

Not Guaranteed:



Since $R(\{S,J,K,B\}) = \{B,C,L\}$, There is no perfect matching.

3. [**3 parts, 2 points each**] Let G be the following bipartite graph.



(a) Find $R(\{x_1, x_3, x_4\})$.

$\{y_1, y_2, y_3, y_4, y_5\}$

(b) What is the deficiency of $\{x_2, x_3, x_4, x_6\}$?

$\delta(S) = |S| - |R(S)| = 4 - |\{y_3, y_5\}| = 4 - 2 = 2$

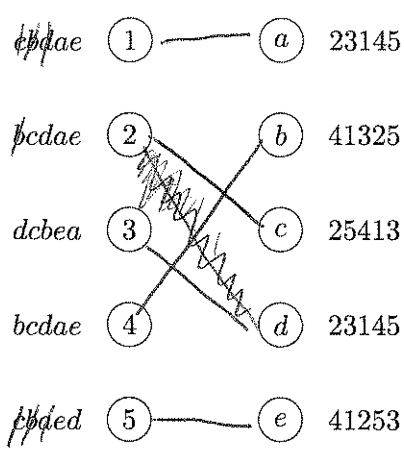
(c) What can you conclude from part (b) about the size of a maximum matching in G?

At least two vertices ~~are~~ in $\{x_2, x_3, x_4, x_6\}$ are unmatched, so a maximum matching has size at most 4.

4. Stable Matchings.

(a) [**7 points**] Given a set $\{1, 2, 3, 4, 5\}$ of men and a set $\{a, b, c, d, e\}$ of women with the following preference lists, find a stable matching.

Men propose:



1 ~~5~~

~~2~~ 4 ~~1~~ ~~5~~

~~1~~ ~~5~~ 2

3 ~~1~~

5

(b) [**3 points**] Which (if any) of the matched pairs are common to all stable matchings? (The preference lists are repeated below.)

Women propose:

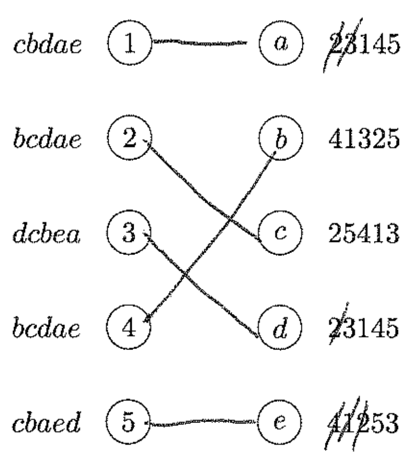
~~c~~ a

~~a~~ c ~~d~~ ~~c~~

~~a~~ d

b ~~c~~

e



All of the matched pairs are common to all stable matchings, because there is only one stable matching.

5. **[4 points]** Is it possible for a stable matching to pair two people who are each others least desirable partners? Either argue that this is impossible or provide an example where this occurs.

Yes:

a b 1   a 1 2

a b 2   b 1 2

In this situation, the only stable matching is $\{1a, 2b\}$, and 2 and b are each others least desirable partners.

6. **[4 parts, 1.5 points each]** Which of the following are groups? Answer yes or no for each; no justification required. Here, $\mathbb{R}$ is the set of real numbers, $\mathbb{R}^+$ is the set of positive real numbers, $+$ denotes standard arithmetic addition, and $\times$ denotes standard arithmetic multiplication.

(a) $(\mathbb{R}, +)$

$\boxed{\text{Yes}}$

(b) $(\mathbb{R}^+, +)$

$\boxed{\text{No}}$; no additive inverses.

(c) $(\mathbb{R}, \times)$

$\boxed{\text{No}}$, because 0 has no multiplicative inverse

(d) $(\mathbb{R}^+, \times)$

$\boxed{\text{Yes}}$

7. **[2 points]** If the received transmission is $r = 00110110$ and the error pattern is $e = 00110011$, what transmission was sent?

$$\begin{array}{rr} r & 00110110 \\ e & 00110011 \\ \hline & \boxed{00000101} \end{array}$$

8. Let $W = \mathbb{Z}_2^5$, so our messages are bitstrings of length 5. Consider the parity bit encoding scheme $E\colon W \to \mathbb{Z}_2^6$ given by $E(x_1 \cdots x_5) = x_1 \cdots x_5 y$ where $y = x_1 + \cdots + x_5 \mod 2$. The decoding function $D(x_1 \cdots x_5 y)$ first checks whether the parity bit y is correct; if it is, then the decoding function returns $x_1 \cdots x_5$ as the transmitted message. Otherwise, the decoding function reports an error. The transmission channel flips bits with probability $p = 0.1$.

   (a) **[2 points]** If a message $x \in W$ is sent *without* any encoding, what is the probability that it is received and decoded properly?

$$(1-p)^5 = (0.9)^5 \approx \boxed{0.59}$$

   (b) **[2 points]** A message is sent using the encoding scheme. If the received message is 011011, what does the decoding function do?

Since $1 = 0+1+1+0+1 \mod 2$, parity bit is correct, and the decoder $\boxed{\text{returns the message } 01101}$.

   (c) **[4 points]** A message is sent using the encoding scheme. There are three possibilities: either the message is correctly decoded, there is an *undetected* error in transmission, or there is a detected error in transmission. Find the probabilities of each of these 3 cases.

Correct: No bit flips - $(1-p)^6 = (0.9)^6 \approx \boxed{0.531}$

Undetected error: 2, 4, or 6 flips:

$$\binom{6}{2} p^2 (1-p)^4 + \binom{6}{4} p^4 (1-p)^2 + \binom{6}{6} p^6 (1-p)^0$$

$$= 15(.1)^2(.9)^4 + 15(.1)^4(.9)^2 + (.1)^6 \approx \boxed{0.0996}$$

Detected Error: 1, 3, or 5 flips: $\binom{6}{1} p(1-p)^5 + \binom{6}{3} p^3(1-p)^3 + \binom{6}{5} p^5(1-p)$

$$= 6(0.1)(0.9)^5 + 20(.1)^3(0.9)^3 + 6(0.1)^5(0.9) \approx \boxed{0.369}$$

9. **[2 points]** List the elements in $S(1001, 1)$.

$\{1001, 0001, 1101, 1011, 1000\}$

10. **[2 points]** Let $x \in \mathbb{Z}_2^{12}$. How many elements are in $S(x, 4)$? You may leave your answer in terms of binomial coefficients.

$\binom{12}{0} + \binom{12}{1} + \binom{12}{2} + \binom{12}{3} + \binom{12}{4}$

11. Consider the encoding function $E\colon \mathbb{Z}_2^2 \to \mathbb{Z}_2^6$ given by

$$E(00) = 000000 \qquad E(01) = 001111$$
$$E(10) = 111100 \qquad E(11) = 110011$$

Minimum distance 4

(a) **[2 points]** If our goal is to detect errors, how many errors can we tolerate?

To detect: $\boxed{\leq 3 \text{ errors}}$

(b) **[1 point]** If our goal is to detect errors and 110111 is received, what should the decoding function do?

Return $\boxed{\text{Error}}$

(c) **[2 points]** If our goal is to correct errors, how many errors can we tolerate?

$2k+1 \leq 4$ | $k \leq 3/2$ $\Rightarrow$ Tolerate $\boxed{\text{at most 1 error.}}$

$2k \leq 3$

(d) **[1 point]** If our goal is to correct errors and 101111 is received, what should the decoding function do?

101111 is within distance 1 of 001111,

So Decoder returns $\boxed{01}$.