Name: _____Solutions_____

Directions: Show all work. Answers without work generally do not earn points. Unless stated otherwise, answers may be left in terms of factorials, permutation numbers, and binomial coefficients. There are 55 points available but the test will be scored out of 50.

1. **[3 points]** A vehicle rental company offers cars in 5 classes (economy, compact, standard, full size, and luxury), and trucks in 3 sizes (small, medium, and large). A customer renting a car may optionally rent a GPS unit and may optionally purchase insurance through the rental company. Customers renting trucks are required to purchase the company's insurance but may decide whether or not to rent a GPS unit. How many ways are there to rent a vehicle from this company?

$$5 \cdot 2 \cdot 2 \quad + \quad 3 \cdot 2 \quad = \quad \boxed{26 \text{ ways}}$$

$\uparrow$ car rentals $\uparrow$ truck rentals

2. **[3 points]** How many 4-digit ATM pins do not contain a repeated digit?

$$10 \cdot 9 \cdot 8 \cdot 7 = 5040$$

3. **[3 points]** How many 4-digit ATM pins have digits that strictly increase from left to right? (So, 0238 and 4789 count, but 5389 and 2234 do not).

$$\boxed{\binom{10}{4}}$$

(Choose 4 numbers, order is determined.)

$$\boxed{= 210.}$$

4. **[3 parts, 3 points each]** Arrangements of words.

   (a) How many ways are there to arrange the letters in the word "LYRICS"?

   $$\boxed{6!} = \boxed{720}$$

   (b) How many ways are there to arrange the letters in the word "LYRICALLY"?

   9 letters; 3 L's; 2 Y's

   $$\boxed{\dfrac{9!}{3! \, 2!}} = \boxed{30,240}$$

   (c) How many ways are there to arrange the letters in the word "LYRICALLY" without consecutive L's?

   ① Arrange YRICAY: $\dfrac{6!}{2!}$

   ② Choose locations for 3 L's: _Y_R_I_C_A_Y_ $\binom{7}{3}$

   $$\boxed{\binom{7}{3} \cdot \dfrac{6!}{2!}} = \boxed{12,600}$$

5. **[3 points]** How many non-negative integral solutions are there to $x_1 + x_2 + x_3 + x_4 + x_5 = 92$?

   ° 92 stars, 4 bars $$\boxed{\binom{96}{4}} = \boxed{3,321,960}$$

6. [**5 points**] A class of n students forms a committee of k students. Of the k students on the committee, 2 are selected as leaders. Give a *combinatorial* argument that $\binom{n}{k}\binom{k}{2} = \binom{n}{2}\binom{n-2}{k-2}$. (An algebraic argument is not sufficient.)

LHS: ① Choose committee $\binom{n}{k}$

② Choose leaders from committee $\binom{k}{2}$
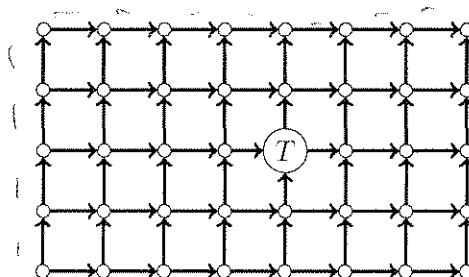
Total #ways = $\boxed{\binom{n}{k}\binom{k}{2}}$

RHS: ① Choose leaders from class $\binom{n}{2}$

② Choose rest of committee $\binom{n-2}{k-2}$

$\binom{n}{2}\binom{n-2}{k-2}$

Therefore $\binom{n}{k}\binom{k}{2} = \binom{n}{2}\binom{n-2}{k-2}$

7. One region of a city consists of a grid of one-way roads, shown below. One of the intersections (labeled T) has a traffic light.

(a) [**3 points**] How many ways are there to travel from the lower-left point to the upper-right point?

4↑   7→ $\boxed{\binom{11}{4}} = \boxed{330}$

(b) [**4 points**] How many ways are there to travel from the lower-left point to the upper-right point *without* passing through the traffic light?

WAYS That DO pass through light:

To Light | Light To corner
2↑, 4→ | 2↑, 3→
$\binom{6}{2}$ e | $\binom{5}{2}$

$= \binom{6}{2}\binom{5}{2}$

Ways that Do not pass through light: $\boxed{\binom{11}{4} - \binom{6}{2}\binom{5}{2}} = \boxed{180}$

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8. Let $A = \{1, \{5\}, 5, 6, 7\}$, $B = \{\varnothing, \{3\}, 4, 5, 6\}$, and $C = \{\varnothing, \{4, 6\}\}$.

   (a) [**6 parts, 1 point each**] True or False? (Write the whole word as your answer.)

       i. $\{6\} \in A$

           FALSE

       ii. $\{\varnothing, \{3\}\} \subseteq B$

           TRUE

       iii. $\{5, \{5\}\} \in A$

           FALSE

       iv. $\{5, \{5\}\} \subseteq A$

           TRUE

       v. $\{4, 6\} \in C$

           TRUE

       vi. $\{4, 6\} \subseteq C$

           FALSE

   (b) [**2 points**] Find $B \cap C$.

   $$B \cap C = \{\varnothing\}$$

   (c) [**2 points**] Find the powerset $\mathcal{P}(C)$.

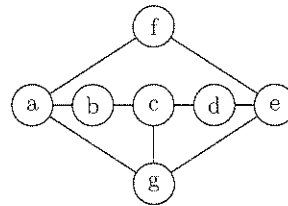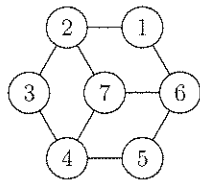 $$\mathcal{P}(C) = \{\varnothing, \{\varnothing\}, \{\{4, 6\}\}, \{\varnothing, \{4, 6\}\}\}$$

9. [**3 points**] Give an example of a connected graph on 6 vertices such that removing any edge results in a disconnected graph.
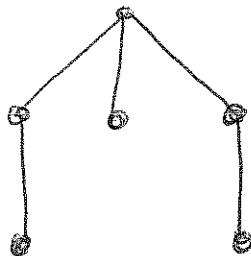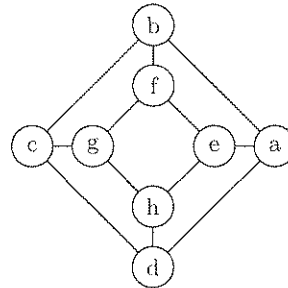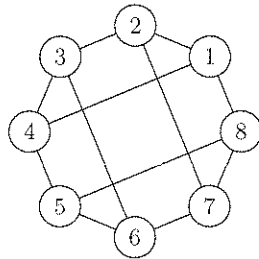

Many other answers are possible --

10. **[3 parts, 3 points each]** For each of the following pairs of graphs, decide whether they are isomorphic. If they are isomorphic, give an isomorphism. Otherwise, argue that they are not isomorphic.
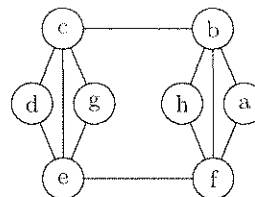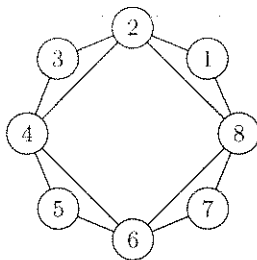
(a)

Isomorphic:

1	2	3	4	5	6	7
a	b	a	d	e	b	g
f	a	b	c	d	e	

(b)

Iso morphic:

1	2	3	4	5	6	7	8
f	b	c	g	h	d	a	e

(c)

Not Isomorphic; Graph on right has triangles that share an edge, but in the graph on the left, the most any two triangles share is a single vertex.