Name: Solutions

Directions: Show all work. Answers without work generally do not earn points. This test has 60 points, but is scored out 50 (scores capped at 50).

1. **[6 points]** Recall that M_{22} is the vector space of (2×2)-matrices. Either show that the following matrices are linearly independent or express the zero vector as a non-trivial linear combination.

$$\begin{bmatrix} 5 & 3 \\ 8 & -1 \end{bmatrix}, \begin{bmatrix} 11 & 5 \\ 12 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 4 & -3 \end{bmatrix}$$

$$a\begin{bmatrix} 5 & 3 \\ 8 & -1 \end{bmatrix} + b\begin{bmatrix} 11 & 5 \\ 12 & 1 \end{bmatrix} + c\begin{bmatrix} -1 & 1 \\ 4 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

rref:
$$\begin{bmatrix} 5 & 11 & -1 \\ 3 & 5 & 1 \\ 8 & 12 & 4 \\ -1 & 1 & -3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 & 3 \\ 5 & 11 & -1 \\ 3 & 5 & 1 \\ 8 & 12 & 4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 & 3 \\ 0 & 16 & -16 \\ 0 & 8 & -8 \\ 0 & 20 & -20 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{array}{c} \begin{matrix} a & b & c \end{matrix} \\ \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}.$$

$\boxed{\text{Not linearly independent}}$ $\begin{bmatrix} -2k \\ k \\ k \end{bmatrix}$

$\boxed{\text{For example } a=-2,\ b=c=1 \text{ gives } \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ as a lin. comb.}}$

2. **[4 parts, 1.5 points each]** Let S be a subset of M_{22} of size n, so that $S = \{v_1, \dots, v_n\}$ where each v_j is a (2×2)-matrix. In (a)–(d) below, you do not need to show any work.

(a) If S is linearly independent, what (if anything) can you say about n?

$$n \le \dim M_{22} = 4$$

(b) If S is linearly dependent, what (if anything) can you say about n?

nothing or $n \ge 1$

(c) If S spans M_{22}, what (if anything) can you say about n?

$$n \ge \dim M_{22} = 4$$

(d) If S does not span M_{22}, what (if anything) can you say about n?

nothing

3. **[2 parts, 3 points each]** Consider P_3, the vector space of polynomials of degree at most 3.

   (a) Find a basis for P_3. What is the dimension of P_3?

   $\boxed{\{1, t, t^2, t^3\}}$ $\qquad \dim P_3 = \boxed{4}$

   (b) Let V be the subspace of P_3 consisting of all polynomials $p(t)$ such that $p(t) = p(-t)$ for every real number t. Find a basis for V. (*Hint:* consider a general element $p(t) = at^3 + bt^2 + ct + d$. What must be true for $p(t) = p(-t)$ to hold?)

$$p(t) = p(-t)$$
$$at^3 + bt^2 + ct + d = a(-t)^3 + b(-t)^2 + c(-t) + d$$
$$at^3 + \cancel{bt^2} + ct + \cancel{d} = -at^3 + \cancel{bt^2} - ct + \cancel{d}$$
$$at^3 + ct = -at^3 - ct$$

Like terms equal:

$\underline{t^3}$: $a = -a \Rightarrow a = 0$

$\underline{t}$: $c = -c \Rightarrow c = 0$

So $V = \{bt^2 + d : b, d \in \mathbb{R}\}$

$\boxed{\text{Basis: } \{1, t^2\}}$

4. **[6 points]** A matrix A and its reduced row-echelon form are displayed below.

$$A = \begin{bmatrix} 1 & 1 & 7 & 1 & 1 & 1 \\ -1 & 1 & 3 & 3 & 0 & -3 \\ 2 & 1 & 9 & 0 & 1 & 4 \\ -2 & 1 & 1 & 4 & 1 & -8 \end{bmatrix} \qquad \operatorname{rref}(A) = \begin{array}{c} \begin{bmatrix} 1 & 0 & 2 & -1 & 0 & 3 \\ 0 & 1 & 5 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{cccccc} & & a & b & & c \end{array} \end{array}$$

Find bases for the row space, the column space, and the null space of A. Clearly label which basis is for which space.

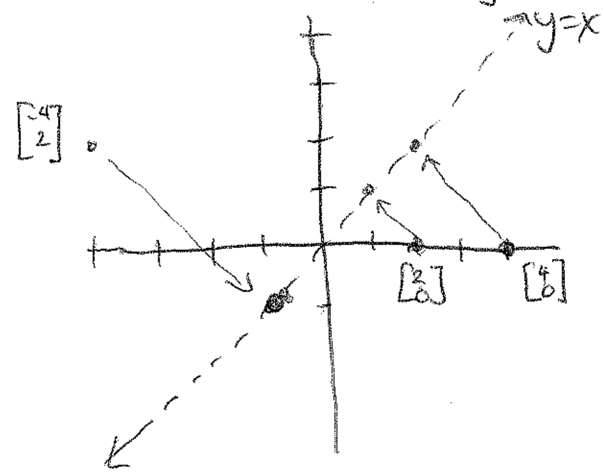
Row Space: $\begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 5 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -2 \end{bmatrix}$ (Row vectors also OK)

Column space: $\begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

Null space: $\begin{bmatrix} -2a + b - 3c \\ -5a - 2b \\ a \\ b \\ 2c \\ c \end{bmatrix} \rightsquigarrow \begin{bmatrix} -2 \\ -5 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}$

5. **[6 points]** Consider the linear transformation $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$. Interpret the transformation f graphically, as a function mapping points in the plane to other points in the plane.

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2}x + \frac{1}{2}y \\ \frac{1}{2}x + \frac{1}{2}y \end{bmatrix}$$



f is the $\boxed{\text{projection}}$ of $\mathbb{R}^2$ onto the line $y=x$

6. **[6 points]** Let $L\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that

$$L\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad L\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad L\left(\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

If possible, compute $L\left(\begin{bmatrix} 3 \\ 9 \\ 5 \end{bmatrix}\right)$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 2 & 1 & 4 & 9 \\ 1 & 0 & 1 & 5 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & -1 & 0 & 3 \\ 0 & -1 & -1 & 2 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 6 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & -1 & -1 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right]. \quad \text{So } \begin{bmatrix} 3 \\ 9 \\ 5 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - 3\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$$

$$\text{So } L\left(\begin{bmatrix} 3 \\ 9 \\ 5 \end{bmatrix}\right) = 4L\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right) - 3L\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) + L\left(\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}\right) = 4\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 3\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}}$$

7. **[2 parts, 6 points each]** Let $L\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by

$$L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = \begin{bmatrix} u_1+u_2 \\ u_2+u_3 \end{bmatrix}.$$

(a) Find a basis for $\ker L$.

$$\begin{bmatrix} u_1+u_2 \\ u_2+u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

rref $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

free var

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

a

$$\begin{bmatrix} a \\ -a \\ a \end{bmatrix}$$

So basis: $\boxed{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}$.

(b) Let S and T be ordered bases for $\mathbb{R}^3$ and $\mathbb{R}^2$ given by

$$S = \left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}}_{v_2}, \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{v_3} \right\} \qquad T = \left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{w_1}, \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{w_2} \right\}$$

Find the matrix A that represents L with respect to S and T.

$$L(v_1) = \begin{bmatrix} 1+0 \\ 0+0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad L(v_2) = \begin{bmatrix} 1+2 \\ 2+0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \qquad L(v_3) = \begin{bmatrix} 1+2 \\ 2+3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$\left[\begin{array}{cc|ccc} 1 & 1 & 1 & 3 & 3 \\ 1 & -1 & 0 & 2 & 5 \end{array}\right] \sim \left[\begin{array}{cc|ccc} 1 & 1 & 1 & 3 & 3 \\ 0 & -2 & -1 & -1 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|ccc} 1 & 1 & 1 & 3 & 3 \\ 0 & 1 & 1/2 & 1/2 & -1 \end{array}\right] \sim \left[\begin{array}{cc|ccc} 1 & 0 & \frac{1}{2} & 5/2 & 4 \\ 0 & 1 & 1/2 & 1/2 & -1 \end{array}\right]$$

So $\boxed{A = \begin{bmatrix} \frac{1}{2} & 5/2 & 4 \\ \frac{1}{2} & \frac{1}{2} & -1 \end{bmatrix}}$.

8. **[2 parts, 6 points each]** Let $L\colon V \to W$ be a linear transformation, let $S = \{v_1, \ldots, v_n\}$ where each v_i is a vector in V, and let $T = \{L(v_1), \ldots, L(v_n)\}$. One of the following statements is true and the other is false.

FALSE $\longrightarrow$ • If S is linearly independent in V, then T is linearly independent in W.

TRUE $\longrightarrow$ • If T is linearly independent in W, then S is linearly independent in V.

(a) Identify the true statement and prove it.

Suppose that $L(v_1), \ldots, L(v_n)$ is lin. indep. in W. We show $v_1, \ldots, v_n$ is lin. indep. in V. Consider a linear combination

$$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = \vec{0}_V$$

Since L is a linear transform, we have that

$$L(a_1 v_1 + \cdots + a_n v_n) = L(\vec{0}_V)$$

$$a_1 L(v_1) + \cdots + a_n L(v_n) = \vec{0}_W.$$

Since $L(v_1), \ldots, L(v_n)$ are lin. indep., it follows that $a_1 = a_2 = \cdots = a_n = 0$.

Therefore $v_1, \ldots, v_n$ are lin. independent. ☒

(b) Identify the false statement and give a counterexample. Your counterexample should consist of a vector space V, a vector space W, a linear transformation $L\colon V \to W$, and appropriate sets S and T.

Let $V = \mathbb{R}^n$ and $W = \{\vec{0}_W\}$ = the zero subspace.

Define $L(v) = \vec{0}_W$ for all $v \in \mathbb{R}^n$.

If $S = \{e_1, \ldots, e_n\}$ (so S is the standard basis), then

$L(e_1), \ldots, L(e_n) = \vec{0}_W, \ldots, \vec{0}_W$.

So S is independent in V but T is linearly dependent.