Name: Solutions

Directions: Show all work. Answers without work generally do not earn points. This test has 60 points.

1. **[2 parts, 6 points each]** Solve the following linear systems.

(a)
$$\begin{aligned} 3x + y &= 5 \\ x - 5y &= 1 \end{aligned}$$

$x - 5y = 1$
$3x + y = 5$

$x - 5y = 1$
$16y = 2$

$y = \frac{1}{8}$

$x = 1 + 5y$
$= 1 + \frac{5}{8}$
$x = 13/8$

$$\boxed{x = 13/8 \quad y = \frac{1}{8}}$$

(b)
$$\begin{aligned} x + 2y - z &= 3 \\ 4x - 6y + 3z &= 2 \\ 6x - 2y + z &= 8 \end{aligned}$$

$x + 2y - z = 3$
$-14y + 7z = -10$
$-14y + 7z = -10$

$x + 2y - z = 3$
$y - \frac{1}{2}z = 5/7$
$0 + 0 = 0$

$(-2)Eq\,2 + Eq\,1 \to Eq\,1$:
$x = \frac{-10}{7} + 3$

$y - \frac{1}{2}z = 5/7$

So $z = k$, $y = 5/7 + \frac{1}{2}k$, and $x = \frac{11}{7}$, for any ~~real~~ a real number z.

2. **[6 parts, 2 points each]** Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 1 \\ -2 & -1 \\ 5 & 0 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & -1 \end{bmatrix}$.

If possible, compute the following. If not possible, write "undefined".

(a) $B^T + C$

$$\begin{bmatrix} 4 & -2 & 5 \\ 1 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & -1 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 5 & 1 & 5 \\ 1 & 1 & -1 \end{bmatrix}}$$

(b) AC

$$\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 7 & -2 \\ -1 & -3 & 0 \end{bmatrix}$$

(c) CA

$$\underbrace{(2\times 3)\quad(2\times 2)}$$

undefined

(d) $A + I_2$

$$\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$$

(e) A^2

$$\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -1 & -2 \end{bmatrix}$$

(f) $B^2 = BB$

$$\underbrace{3\times 2 \quad 3\times 2}$$

undefined

3. **[6 points]** Prove that if $A\mathbf{x} = \mathbf{b}$ has more than one solution, then it has infinitely many solutions. (Hint: if $\mathbf{x_1}$ and $\mathbf{x_2}$ distinct solutions, show that appropriate linear combinations of $\mathbf{x_1}$ and $\mathbf{x_2}$ yield infinitely many solutions.)

Suppose that $Ax_1 = b$ and $Ax_2 = b$. We show that

$rx_1 + sx_2$ is also a solution when $r+s=1$.

We have that

$$A(rx_1 + sx_2) = A(rx_1) + A(sx_2) = rAx_1 + sAx_2 = rb + sb = (r+s)b = b.$$

Therefore $Ax=b$ has infinitely many solutions.

4. **[6 points]** Let A and B be matrices of appropriate sizes. One statement is true and the other is false. Select the true statement and prove it.

TRUE ⇒ • If the rth row of A is all zeros, then the rth row of AB is all zeros.
- If the rth column of A is all zeros, then the rth column of AB is all zeros.

Let A be an $(m \times n)$-matrix and B be an $(n \times p)$-matrix. Let $C = AB$.
Since the r^{th} row of A is all zeros, we have $a_{rj} = 0$ for all j. We show that $c_{rj} = 0$ for all j.

By the definition of matrix multiplication,

$$c_{rj} = \sum_{k=1}^{n} a_{rk} b_{kj} = \sum_{k=1}^{n} 0 \cdot b_{kj} = 0.$$

Therefore all entries in the r^{th} row of C are zeros.

5. **[6 points]** Give an example of a lower-triangular (3×3)-matrix with as many non-zero entries as possible.

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix}$$

6. **[6 points]** Let $A = \begin{bmatrix} 4 & 2 & -3 & 6 \\ 1 & 0 & 2 & 0 \\ 7 & 1 & 3 & -4 \end{bmatrix}$. Find a matrix in reduced row echelon form that is row-equivalent to A.

$$\sim \begin{bmatrix} 1 & 0 & 2 & 0 \\ 4 & 2 & -3 & 6 \\ 7 & 1 & 3 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 2 & -11 & 6 \\ 0 & 1 & -11 & -4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -11 & -4 \\ 0 & 2 & -11 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -11 & -4 \\ 0 & 0 & 11 & 14 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -11 & -4 \\ 0 & 0 & 1 & \frac{14}{11} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & \frac{-28}{11} \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 14/11 \end{bmatrix}$$

7. **[6 points]** Let $A = \begin{bmatrix} 3 & 6 \\ -1 & 2 \end{bmatrix}$. Find elementary matrices $E_1, \ldots, E_4$ such that $E_4E_3E_2E_1A = I_2$.

$\underline{E_1}$:
$$\begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}$$

$\underline{E_2}$:
$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$$

$\underline{E_3}$:
$$\begin{bmatrix} 1 & 0 \\ 0 & 1/4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$\underline{E_4}$:
$$\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\underline{\text{So}}$:

$$\boxed{\underbrace{\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}}_{E_4} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{bmatrix}}_{E_3} \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}_{E_2} \underbrace{\begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix}}_{E_1} A = I_2}$$

8. **[6 points]** Let $A = \begin{bmatrix} 2 & -1 & 8 \\ 1 & 0 & 2 \\ -2 & 1 & 0 \end{bmatrix}$. If it exists, find A^{-1}. Otherwise, write "does not exist".

$$\left[\begin{array}{ccc|ccc} 2 & -1 & 8 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ -2 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 2 & -1 & 8 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & -1 & 4 & 1 & -2 & 0 \\ 0 & 1 & 4 & 0 & 2 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & -4 & -1 & 2 & 0 \\ 0 & 1 & 4 & 0 & 2 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & -4 & -1 & 2 & 0 \\ 0 & 0 & 8 & 1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & -4 & -1 & 2 & 0 \\ 0 & 0 & 1 & 1/8 & 0 & \frac{1}{8} \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2/8 & 1 & \frac{-2}{8} \\ 0 & 1 & 0 & -4/8 & 2 & +\frac{4}{8} \\ 0 & 0 & 1 & \frac{1}{8} & 0 & \frac{1}{8} \end{array}\right] \qquad A^{-1} = \begin{bmatrix} -1/4 & 1 & -1/4 \\ -\frac{1}{2} & 2 & 1/2 \\ \frac{1}{8} & 0 & \frac{1}{8} \end{bmatrix}$$

$$\text{or } A^{-1} = \frac{1}{8} \begin{bmatrix} -2 & 8 & -2 \\ -4 & 16 & +4 \\ 1 & 0 & 1 \end{bmatrix}$$