Name: Key

This test has 6 pages, each worth 10 points. The test is scored out of 50 points. Your lowest scoring page is dropped.

1. [**2 points**] What conclusion(s), if any, can be reached from the following given hypotheses? If Sam does not rush, then Sam will be late. If Sam is late, then Sam will be fired. Sam is not late.

   Sam rushes.

   No conclusion about whether Sam is fired.

2. [**5 parts, 1 point each**] Using the given statement letters, translate the following sentences into wffs.

C: The cat looks guilty	A: The cat eats the bird	S: The bird sings
M: The owner is mad at the cat	P: The cat purrs	E: The bird escapes

   (a) The bird escapes or the cat eats the bird.

   $E \vee A$

   (b) If the owner is mad at the cat, then the cat does not purr.

   $M \to P'$

   (c) The cat looks guilty if and only if it eats the bird.

   $C \leftrightarrow A$

   (d) The owner is mad at the cat, and the bird sings only if it escapes.

   $M \wedge (S \to E)$

   (e) The cat purrs unless the owner is mad at the cat.

   $M' \to P$ or $P' \to M$ or $P \vee M$

3. [**3 parts, 1 point each**] Write the negation of the following statements using simple and natural English sentences.

   (a) The bird escapes or the cat eats the bird.

   The bird does not escape and the cat does not eat the bird

   (b) If the owner is mad at the cat, then the cat does not purr.

   The owner is mad at the cat and the cat purrs.

   (c) The cat looks guilty if and only if it eats the bird.

   The cat looks guilty or the cat eats the bird, but not both.

   OR: The cat looks guilty if and only if it does not eat the bird.

4. Two parts.

   (a) [**4 points**] Write a truth table for the following wff:

$$((A \to B) \leftrightarrow (B \to A))'$$

A	B	$A \to B$	$B \to A$	$(A\to B)\leftrightarrow(B\to A)$	$((A\to B)\leftrightarrow(B\to A))'$
T	T	T	T	T	F
T	F	F	T	F	T
F	T	T	F	F	T
F	F	T	T	T	F

   (b) [**1 point**] Is the wff a tautology, a contradiction, or neither? Neither

5. [**5 points**] Prove that the following wff is a tautology by giving a proof sequence.

Derivation Rule	Name/Abbreviation for Rule
$P \vee Q \iff Q \vee P$ $P \wedge Q \iff Q \wedge P$	Commutative—comm
$(P \vee Q) \vee R \iff P \vee (Q \vee R)$ $(P \wedge Q) \wedge R \iff P \wedge (Q \wedge R)$	Associative—ass
$(P \vee Q)' \iff P' \wedge Q'$ $(P \wedge Q)' \iff P' \vee Q'$	De Morgan's laws—De Morgan
$P \to Q \iff P' \vee Q$	Implication—imp
$P \iff (P')'$	Double negation—dn
$P \leftrightarrow Q \iff (P \to Q) \wedge (Q \to P)$	Defn of Equivalence—equ
$\left.\begin{array}{r} P \\ P \to Q \end{array}\right\} \implies Q$	Modus ponens—mp
$\left.\begin{array}{r} P \to Q \\ Q' \end{array}\right\} \implies P'$	Modus tollens—mt
$\left.\begin{array}{r} P \\ Q \end{array}\right\} \implies P \wedge Q$	Conjunction—con
$P \wedge Q \implies \left\{\begin{array}{l} P \\ Q \end{array}\right.$	Simplification—sim
$P \implies P \vee Q$	Addition—add

$$((B \wedge C) \to A) \wedge C \wedge A' \to B'$$

Column 1:		Column 2:	
1. $(B \wedge C) \to A$	hyp	5. $B \to C'$	4 imp
2. A'	hyp	6. C	hyp
3. $(B \wedge C)'$	1,2 mt	7. $(C')'$	6 dn
4. $B' \vee C'$	3 DeMorgan	8. B'	5,7 mt

6. [**6 parts, 1 point each**] Using the given predicates, translate the following sentences into wffs. The domain is all people in the world.

$O(x,y)$: x is at least as old as y	$M(x)$: x is married	$W(x)$: x is a woman
$F(x,y)$: x and y are friends	$R(x)$: x is rich	$S(x)$: x is a student

(a) There is a rich student.

$$\exists x[R(x) \wedge S(x)]$$

(b) There are some rich people who are friends with everyone.

$$\exists x[R(x) \wedge \forall y[F(x,y)]]$$

(c) Every woman has a younger friend.

$$\forall x[W(x) \rightarrow \exists y[O(x,y) \wedge F(x,y)]]$$

(d) Some students are only friends with unmarried people.

$$\exists x[S(x) \wedge \forall y[F(x,y) \rightarrow M(x)']]$$

(e) Only women are friends with married students.

$$\forall x[\exists y[F(x,y) \wedge M(y) \wedge S(y)] \rightarrow W(x)]$$

(f) The oldest person in the world is not rich.

$$\exists x[\forall y[O(x,y)] \wedge R(x)']$$

7. [**4 parts, 1 point each**] Write the negation of the following statements using simple and natural English sentences.

(a) There are some rich people who are friends with everyone.

Every rich person is (not friends) with someone.

(b) Every woman has a younger friend.

Some women have only older friends.

(c) Some students are only friends with unmarried people.

Every student has a married friend.

(d) Only women are friends with married students.

Some man is friends with a married student.

8. **[5 parts, 2 points each]** Determine whether the following sentences are valid. If the sentence is not valid, give an interpretation under which the sentence is false. Clearly indicate the domain of any interpretation you give.

   (a) $\forall x\,[Q(x)] \vee \forall x\,[Q(x)']$

   Not valid.

   Domain = integers

   $Q(x)$: x is even

   (b) $\left(\forall x\,[P(x) \to Q(x)] \wedge \forall x\,[P(x)]\right) \to \forall x\,[Q(x)]$

   Valid

   (c) $(\forall x\,[P(x)] \to \forall x\,[Q(x)]) \to \forall x\,[P(x) \to Q(x)]$

   Not Valid

   Domain = integers

   $P(x)$: x is odd $Q(x)$: x is even

   (d) $\forall x\,[P(x) \vee Q(x)] \wedge \exists x\,[Q(x)] \to \exists x\,[P(x)]$

   Not valid. Domain = integers

   $P(x)$: $x = \pi$

   $Q(x)$: x is an integer

   (e) $\forall x\,[\exists y\,[P(x,y)]] \to \forall y\,[\exists x\,[P(x,y)]]$

   Not Valid. Domain = $\{1, 2, 3, \ldots\}$

   $P(x,y)$: $x < y$

9. **[5 points]** Give a formal proof (i.e. proof sequence) that the following formula is valid.

$$\forall x\,[Q(x) \vee \forall y\,[P(x,y)]] \wedge \exists x\,[Q(x)'] \to \exists x\,[\forall y\,[P(x,y)]]$$

1. $\exists x\,[Q(x)']$ hyp
2. $Q(a)'$ 1, ei
3. $\forall x\,[Q(x) \vee \forall y\,[P(x,y)]]$ hyp
4. $Q(a) \vee \forall y\,[P(a,y)]$ 3, ui
5. $(Q(a)')' \vee \forall y\,[P(a,y)]$ 4, dn
6. $Q(a)' \to \forall y\,[P(a,y)]$ 5, imp
7. $\forall y\,[P(a,y)]$ 2, 6 mp
8. $\exists x\,[\forall y\,[P(x,y)]]$ 7, eg

10. **[5 points]** Prove that the sum of two odd integers is even.

Proof Let x and y be odd integers. We prove that $x+y$ is even. Since x and y are odd, $x = 2n+1$ and $y = 2m+1$ for some integers m and n. Therefore

$$x+y = (2n+1)+(2m+1) = 2(m+n+1)$$

and it follows that $x+y$ is even. ∎

11. [**5 points**] Prove that the sum of two rationals is rational.

Proof Let x and y be rational numbers. We show that $x+y$ is rational. Since x and y are rational, $x = \frac{a}{b}$ and $y = \frac{c}{d}$ for some integers a, b, c, d where $b \neq 0$ and $d \neq 0$.

Therefore $x+y = \frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{cb}{bd} = \frac{ad+cb}{bd}$.

Since $ad+cb$ and bd are integers with $bd \neq 0$, it follows that $x+y$ is rational.

12. [**5 points**] Prove that an integer n is even if and only if n^2 is even.

Proof: $(\rightarrow)$ We show if n is even, then n^2 is even. Let n be an even integer. Since n is even, $n = 2r$ for some integer r. Hence $n^2 = (2r)^2 = 4r^2 = 2(2r^2)$ and therefore n^2 is even.

$(\leftarrow)$ Next, we show that if n^2 is even, then n is even. To do so, we prove the contrapositive: if n is odd, then n^2 is odd. Let n be an odd integer; hence $n = 2s+1$ for some integer s. Therefore

$$n^2 = (2s+1)^2 = 4s^2 + 4s + 1 = 2(2s^2+2s)+1,$$

which implies that n^2 is odd.