

# CSTBC Exam 3

## Due: August 13, 2007, 11:59pm

August 14, 2007

This exam is open notes/open lecture and covers material from lectures 17-24. You are welcome to use any of the course material linked from the CSTBC website. You should not use other reference materials. Please send me your solutions to the exam via email. If you have any questions, please ask me.

### 1 Balls in Bins

Suppose that  $m$  balls are thrown into  $n$  bins uniformly and independently at random. In terms of  $m$  and  $n$ , what is the expected number of bins that contain at least one ball?

### 2 A Hat Game

A group of three people play the following game. Each person is blindfolded and given a hat to wear. The hats are one of two colors (red or blue), and the colors are assigned independently and uniformly at random. Once each person is wearing a hat, the blindfolds are removed and each person can see the other two hats, but is unable to see his/her own hat. Without communicating in any way, each person simultaneously writes down one of three responses on a sheet of paper: "I guess my hat is red", "I guess my hat is blue", or "I choose not to guess".

The group wins if it is the case that no one guesses incorrectly and at least one person guesses correctly. Describe a way for the group to play the game so that they win with probability  $3/4$ . (The group is allowed to discuss their strategy before the game begins, but once the game begins, no communication is allowed.)

### 3 Interviewing Candidates

A manager at a software company must hire a new programmer. The manager has the resumes of  $n$  people who applied for the job and schedules interviews as follows. First, the manager shuffles all the resumes so that they are in a random order. Next, the manager looks at each resume in order and schedules an interview whenever the current applicant's resume is superior to all previous resumes. What is the expected number of interviews that the manager schedules?

### 4 A Random Walk

In Lecture 23, we introduce the concept of a random walk on a line of  $n$  spaces. In that case, the token moves to the left with probability  $1/2$  and to the right with probability  $1/2$ . Suppose instead that the token moves to the left with probability  $1/3$  and to the right with probability  $2/3$ . Now what is the probability that we win (i.e. the token falls off of the right hand side)?

### 5 More Pirates

In Exam 2, we had the following problem: "After distributing their treasure, our  $n$  pirates from Exam 1 have worked up an appetite. The pirate ship's cafeteria offers three different, non-overlapping dinner times. As the strongest pirate, you are in charge of assigning each pirate to one of the three dinner slots. Unfortunately, not all pirates get along with each other. You have a list of  $k$  pairs of pirates that have fought each other in the past. Prove that you can assign pirates to dinners so that at most  $k/3$  of the troublesome pairs eat dinner at the same time."

Suppose that  $n$  is divisible by three and that the ship's cafeteria has limited seating, so that each dinner can accommodate only  $n/3$  pirates. Prove that even with this added restriction, you can assign the pirates to dinner slots so that at most  $k/3$  troublesome pairs eat dinner at the same time. [Hint: although an inductive proof is possible, the probabilistic method (see Lecture 24) is easier.]