Graph 2-rankings

Jordan Almeter[*], Samet Demircan[†], Andrew Kallmeyer[‡], Kevin G. Milans[§], Robert Winslow[¶]

August 13, 2016

Abstract

A 2-*ranking* of a graph G is an ordered partition of the vertices of G into independent sets $V_1, \ldots, V_t$ such that for $i < j$, the subgraph of G induced by $V_i \cup V_j$ is a star forest in which each vertex in V_i has degree at most 1. A 2-ranking is intermediate in strength between a star coloring and a distance-2 coloring. The 2-*ranking number of* G, denoted $\chi_2(G)$, is the minimum number of parts needed for a 2-ranking.

For the d-dimensional cube Q_d, we prove that $\chi_2(Q_d) = d + 1$. As a corollary, we improve the upper bound on the star chromatic number of products of cycles when each cycle has length divisible by 4.

Let $\chi'_2(G) = \chi_2(L(G))$, where $L(G)$ is the line graph of G; equivalently, $\chi'_2(G)$ is the minimum t such that there is an ordered partition of $E(G)$ into t matchings $M_1, \ldots, M_t$ such that for each j, the matching M_j is induced in the subgraph of G with edge set $M_1 \cup \cdots \cup M_j$. We show that $\chi'_2(K_{m,n}) = nH_m$ when $m!$ divides n, where $K_{m,n}$ is the complete bipartite graph with parts of sizes m and n, and H_m is the harmonic sum $1 + \cdots + \frac{1}{m}$. We also prove that $\chi_2(G) \le 7$ when G is subcubic and show the existence of a graph G with maximum degree k and $\chi_2(G) \ge \Omega(k^2/\log(k))$.

1 Introduction

A path consisting of a single vertex is *trivial*; paths with positive length are *nontrivial.* In a graph whose vertices are assigned integer ranks, a path is *well-ranked* if its endpoints have distinct ranks or some interior vertex has a higher rank than the endpoints. A *ranking* of a graph G is an assignment of ranks to $V(G)$ such that every nontrivial path is well-ranked. Graph rankings have arisen in mathematics and computer science; see the section on rankings in Gallian's dynamic survey [5] for a summary of results and background. A k-*ranking* is a relaxation in which each nontrivial path of length at most k is well-ranked. The k-*ranking number* of G, denoted $\chi_k(G)$, is the minimum number of ranks in a k-ranking of G.

Graph k-rankings were introduced by Karpas, Neiman, and Smorodinsky [6], who used the term *unique-superior coloring* for the case $k = 2$. In our terminology, Karpas, Neiman, and Smorodinsky proved that the maximum, over all n-vertex trees T, of $\chi_2(T)$ is $\Theta(\frac{\log n}{\log \log n})$. Trees are K_3-minor-free; it turns out that the k-ranking number of a graph grows at most logarithmically when some

[*]College of William and Mary
[†]West Virginia University
[‡]Miami University
[§]West Virginia University
[¶]University of Kansas

minor is excluded. Specifically, Karpas, Neiman, and Smorodinsky show that for each graph H, there is a constant s such that each n-vertex H-minor-free graph G satisfies $\chi_k(G) \le s(k+1)\log n$. A graph G is *d-degenerate* if each subgraph of G has a vertex of degree at most d. They also prove that each n-vertex d-degenerate graph G satisfies $\chi_2(G) \le d(4\sqrt{n}+1)$ and construct n-vertex 2-degenerate graphs G with $\chi_2(G) > n^{1/3}$.

Graph 2-rankings are intermediate in strength between *star colorings*, where the vertices of a graph are partitioned into independent sets with each pair of parts inducing a star forest, and *distance 2-colorings*, where the vertices of a graph are partitioned into independent sets with each pair of parts inducing a graph with maximum degree at most 1. A 2-ranking of a graph G interpolates between these by giving an ordered partition of $V(G)$ into independent sets $V_1, \dots, V_t$ such that for $i < j$, the subgraph of G induced by $V_i \cup V_j$ is a star forest in which each vertex in V_i has degree at most 1. Consequently, $\chi_s(G) \le \chi_2(G) \le \chi(G^2)$, where $\chi_s(G)$ is the *star chromatic number* of G and $\chi(G^2)$ is the usual chromatic number of the graph obtained from G by joining vertices at distance 2.

For the d-dimensional cube Q_d, Fertin, Raspaud, and Reed [4] proved $(d+3)/2 \le \chi_s(Q_d) \le d+1$. Wan [9] proved that $\chi(Q_d^2) = d+1$ when d is one less than a power of two, and it is easy to see that $\chi(Q_d^2) > d+1$ when d does not have this form. In Section 2, we extend a classical linear algebra technique to show that $\chi_2(Q_d) = d+1$ for all d. A graph is *toroidal* if it is the cartesian product of cycles. As a corollary, if G is toroidal graph with d factor cycles, each having length divisible by 4, then $\chi_2(G) = 2d+1$. Some assumptions on the cycle lengths are necessary, since in Section 6, we show that $\chi_2(G) = 6 > 2d+1$ when G is the product of C_3 and a large odd cycle. The corollary has implications for the star chromatic number of certain toroidal graphs. Pór and Wood [8] proved $\chi_s(G) \le 6d + O(\log d)$ when G is toroidal with d factor cycles (with no restriction on the factor lengths). Earlier, Fertin, Raspaud, and Reed [4] proved that $\chi_s(G) \le 2d+1$ when G is the product of d cycles, each of whose lengths is divisible by $2d+1$ (and that $\chi_s(G) \le 2d^2+d+1$ in the general case). When each factor cycle has length divisible by 4, our corollary gives $\chi_s(G) \le \chi_2(G) = 2d+1$, improving the upper bound on the star chromatic number in this case.

The *line graph* of a graph G, denoted $L(G)$, is the graph with vertex set $E(G)$ where e_1 and e_2 are adjacent in $L(G)$ if and only if e_1 and e_2 share a common endpoint in G. Let $\chi_2'(G) = \chi_2(L(G))$. In terms of G, a 2-ranking of $L(G)$ is an ordered partition of $E(G)$ into matchings $M_1, \dots, M_t$ such that for each j, the matching M_j is induced in the subgraph of G with edge set $M_1 \cup \dots \cup M_j$. In Section 3, we study $\chi_2'(K_{m,n})$, where $K_{m,n}$ is the complete bipartite graph with parts of sizes m and n. When $m!$ divides n, we obtain an exact result: $\chi_2'(K_{m,n}) = nH_m$, where H_m is the harmonic sum $1 + \frac{1}{2} + \dots + \frac{1}{m}$. For each fixed m, it follows that $\chi_2'(K_{m,n}) = (1+o(1))n\ln m$ as $n \to \infty$. For the diagonal case, we obtain only $\Omega(n \log n) \le \chi_2'(K_{n,n}) \le O(n^{\log_2 3})$. It would be interesting to find the order of growth of $\chi_2'(K_n)$ and $\chi_2'(K_{n,n})$.

Problem 1. *Determine the order of growth of $\chi_2'(K_n)$ and $\chi_2'(K_{n,n})$.*

It is easy to see that if G has maximum degree k, then $\chi_2(G) \le \chi(G^2) \le \Delta(G^2)+1 \le k^2+1$. In Section 4, we adapt a probabilistic construction of Fertin, Raspaud, and Reed [4] to obtain graphs with maximum degree k and 2-ranking number $\Omega(k^2/\log k)$. In Section 5, we show that subcubic graphs have 2-ranking number at most 7, and conjecture that aside from a single exception, subcubic graphs have 2-ranking number at most 5.

2 The hypercube

The *d-dimensional cube*, denoted Q_d, is the graph with vertex set $\{0,1\}^d$ where u and v are adjacent if u and v differ in exactly one coordinate. We prove that $\chi_2(Q_d) = d+1$. The lower bound follows from a useful proposition. A graph is *k-degenerate* if every subgraph contains a vertex of degree at most k. The *degeneracy* of a graph G is the minimum integer k such that G is k-degenerate.

Proposition 1. *If G is a graph with degeneracy k, then $\chi_2(G) \geq k+1$.*

Proof. Since G is not $(k-1)$-degenerate, G contains a subgraph H with minimum degree at least k. Consider a 2-ranking of G, and let v be a vertex of minimum rank in H. The ranks of the neighbors of v in H are distinct, and the rank of v differs from all of these. It follows that $\chi_2(G) \geq k+1$. □

Since Q_d is d-regular, it follows that $\chi_2(Q_d) \geq d+1$. Wan [9] proved that $\chi(Q_d^2) = d+1$ when $d = 2^k - 1$ for some integer k, and it follows that $d+1 \leq \chi_2(Q_d) \leq \chi(Q_d^2) = d+1$ in this case. Each color class in a proper coloring of Q_d^2 has size at most $\lfloor 2^d/(d+1) \rfloor$, and it follows that $\chi(Q_d^2) \geq 2^d / \lfloor 2^d/(d+1) \rfloor$. Therefore $\chi(Q_d^2) > d+1$ when d does not have the form $2^k - 1$. Nonetheless, we show that $\chi_2(Q_d) = d+1$ for all d. Although determining the exact value of $\chi(Q_d^2)$ remains open, Östergård [7] proved that $\chi(Q_d^2) = (1+o(1))d$.

We view the vertex set of Q_d as $\mathbb{F}_2^d$, the d-dimensional vector space over the finite field $\mathbb{F}_2$ with 2 elements. For $u \in \mathbb{F}_2^d$, we define the *support* of u to be the set of coordinates in $[d]$ where u has value 1. The *weight* of u, denoted $w(u)$, is the size of the support of u. Note that for all vertices $u, v \in \mathbb{F}_2^n$, we have that $\mathrm{dist}(u,v) = w(u-v)$, where $\mathrm{dist}(u,v)$ is the length of a shortest path from u to v in Q_d. For integers i and j, we use $[i,j]$ to denote the interval $\{i, i+1, \ldots, j\}$.

Theorem 2. $\chi_2(Q_d) = d+1$.

Proof. As we have seen, $\chi_2(Q_d) \geq d+1$. We prove the upper bound by induction on d. The result for $d \in \{0,1\}$ is trivial, since the vertices may be assigned distinct ranks in the interval $[0,d]$. Suppose that $d \geq 2$, and express d as $t + 2^k$ where $k \geq 1$ and $0 \leq t \leq 2^k - 1$. Given $u \in \mathbb{F}_2^d$, we let u^- be the vector in $\mathbb{F}_2^t$ consisting of the first t coordinates of u and we let u^+ be the 2^k-dimensional vector consisting of the remaining coordinates. If $w(u^+)$ is even, then we set the rank of u equal to the rank in $[0,t]$ assigned to u^- inductively.

If $w(u^+)$ is odd, then we assign u a rank in the interval $[t+1,d]$ as follows. Let A be a $(k \times d)$-matrix whose first t columns are distinct, nonzero vectors in $\mathbb{F}_2^k$ and whose last 2^k columns form a permutation of $\mathbb{F}_2^k$. Let $\phi\colon \mathbb{F}_2^k \to [t+1,d]$ be a bijection. When $w(u^+)$ is odd, we set the rank of u to be $\phi(Au)$. Ranks in the range $[0,t]$ are *low*, and ranks in the range $[t+1,d]$ are *high*.

We show that this assignment is a 2-ranking. Let P be a uv-path of length 1 or 2. Note that $w(u-v)$ equals the length of P. We prove that P is well-ranked by examining several cases.

Case 1. *The support of $u-v$ is contained in the first t coordinates.*

We have that $u^+ = v^+$. If $w(u^+)$ and $w(v^+)$ are even, then the vertices of P are colored inductively and so P is well-ranked by induction. Otherwise both $w(u^+)$ and $w(v^+)$ are odd, and so u is assigned rank $\phi(Au)$ and v is assigned rank $\phi(Av)$. Since $w(u-v) \in \{1,2\}$, it follows that $A(u-v)$ is the sum of one or two of the first t columns of A. Since these columns are nonzero and distinct, we have that $A(u-v) \neq 0$ and it follows that u and v are assigned different ranks. Therefore P is well-ranked.

Case 2. *The support of* $u - v$ *is contained in the last* 2^k *coordinates and* $\text{dist}(u, v) = 1$.

Since $w(u - v) = \text{dist}(u, v) = 1$, it follows that $w(u^+)$ and $w(v^+)$ have opposite parity, implying that one of $\{u, v\}$ is assigned a high rank and the other is assigned a low rank.

Case 3. *The support of* $u - v$ *is contained in the last* 2^k *coordinates and* $\text{dist}(u, v) = 2$.

We have that $w(u^+)$ and $w(v^+)$ have the same parity. Let x be the internal vertex on P, and note that $w(x^+)$ has opposite parity. If both $w(u^+)$ and $w(v^+)$ are even and $w(x^+)$ is odd, then the endpoints u and v are assigned low rank while x is assigned high rank, and so P is well-ranked. If both $w(u^+)$ and $w(v^+)$ are odd, then u has rank $\phi(Au)$ and v has rank $\phi(Av)$. Since $w(u - v) = \text{dist}(u, v) = 2$ and the support of $u - v$ is contained in the last 2^k coordinates, it follows that $A(u - v)$ is the sum of two columns from the last 2^k columns in A. Since these are distinct, it follows that $A(u - v) \neq 0$. Therefore $Au \neq Av$, and so P is well-ranked.

Case 4. *The support of* $u - v$ *intersects both the first* t *coordinates and the last* 2^k *coordinates.*

We have that $w(u^+)$ and $w(v^+)$ have opposite parity. Therefore one of $\{u, v\}$ has high rank and the other has low rank.

In all cases, P is well-ranked. □

The 2-ranking given in Theorem 2 assigns the same low rank to u and v whenever $u^- = v^-$ and both $w(u^+)$ and $w(v^+)$ are even. Consequently, when $d \geq 3$, many pairs of vertices at distance 2 share a common rank. When d is one less than a power of two, a proper coloring of Q_d^2 is a 2-ranking of Q_d in which pairs of vertices at distance 2 receive distinct ranks. It follows that when $d \geq 3$ and d has the form $2^k - 1$, there are non-isomorphic optimal 2-rankings of Q_d. The situation when d has the form 2^k may be different. For $d \in \{1, 2\}$, there is only one optimal 2-ranking of Q_d up to isomorphism. We suspect that Q_4 has only one optimal 2-ranking up to isomorphism. Is it true that Q_d has one optimal 2-ranking up to isomorphism when d is a power of two?

The *cartesian product* of G and H, denoted $G \,\square\, H$, is the graph with vertex set $V(G) \times V(H)$ where (u, v) is adjacent to (u', v') if and only if $u = u'$ and $vv' \in E(H)$ or $uu' \in E(G)$ and $v = v'$.

Corollary 3. *If* G *is the cartesian product of* d *cycles, each of which has length divisible by* 4*, then* $\chi_s(G) \leq \chi_2(G) = 2d + 1$.

Proof. Since G has degeneracy $2d$, Proposition 1 implies that $\chi_2(G) \geq 2d + 1$. Note that Q_{2d} is the cartesian product of d copies of C_4. Viewing $V(Q_{2d})$ as $\mathbb{Z}_4^d$, let $f \colon \mathbb{Z}_4^d \to [2d+1]$ be a 2-ranking of Q_{2d}. We use f to color G. Let $m_1, \ldots, m_d$ be the cycle lengths of the factors of G, and view $V(G)$ as $\{(x_1, \ldots, x_d) \colon x_i \in \mathbb{Z}_{m_i}\}$. For $x \in V(G)$, let x' be the vertex in Q_{2d} obtained from x by reducing each coordinate of x modulo 4. We assign $x \in V(G)$ the rank $f(x')$. Since each path in G of length at most 3 maps to a path in Q_{2d} of the same length whose vertices are assigned the same ranks as in G, it follows that G inherits the 2-ranking of Q_{2d}. □

Let G be the cartesian product of d cycles. Fertin, Raspaud, and Reed [4] proved that $d + 2 \leq \chi_s(G) \leq 2d^2 + d + 1$, and improved the upper bound to $2d + 1$ in the case that $2d + 1$ divides the length of each factor cycle. Pór and Wood [8] proved that G admits a proper $(6d + O(\log d))$-coloring in which each pair of color classes induces a matching and isolated vertices; their result directly implies that $\chi_s(G) \leq 6d + O(\log d)$. Corollary 3 extends the divisibility conditions under which it is known that $\chi_s(G) \leq 2d + 1$.

3 Cartesian products of complete graphs

Recall that $\chi_2'(G) = \chi_2(L(G))$, where $L(G)$ is the line graph of G. In this section, we study $\chi_2'(K_{m,n})$, or, equivalently, $\chi_2(K_m \,\square\, K_n)$. For each fixed m, we obtain $\chi_2(K_m \,\square\, K_n)$ asymptotically. When $m = n$, our bounds are far apart. A 2-ranking of $K_m \,\square\, K_n$ can be viewed as an $(m \times n)$-matrix A such that $A(i,j)$ is the rank of $(u_i, v_j) \in V(K_m \,\square\, K_n)$. The condition that paths of length 1 are well-ranked is equivalent to the rows and columns of A having distinct entries. The condition that paths of length 2 are well-ranked is equivalent to the property that $A(i,j) = A(i',j')$ implies that the opposite corners $A(i,j')$ and $A(i',j)$ are larger than $A(i,j)$ and $A(i',j')$.

For positive integers a, b, c, d, our first result obtains a 2-ranking of $K_{ac} \,\square\, K_{bd}$ from 2-rankings of $K_a \,\square\, K_b$ and $K_c \,\square\, K_d$.

Proposition 4. $\chi_2(K_{ac} \,\square\, K_{bd}) \le \chi_2(K_a \,\square\, K_b) \cdot \chi_2(K_c \,\square\, K_d)$.

Proof. Let $k = \chi_2(K_a \,\square\, K_b)$ and $\ell = \chi_2(K_c \,\square\, K_d)$. Let A be an $(a \times b)$-matrix with entries in $\{0, \ldots, k-1\}$ encoding an optimal 2-ranking of $K_a \,\square\, K_b$, and let B be an $(c \times d)$-matrix with entries in $\{0, \ldots, \ell - 1\}$ encoding an optimal 2-ranking of $K_c \,\square\, K_d$. We use block operations to construct a 2-ranking of $K_{ac} \,\square\, K_{bd}$.

Let C be the $(ac \times bd)$-matrix obtained from A and B by replacing each entry $A(i,j)$ in A with the $(c \times d)$-matrix $\ell A(i,j) + B$. It is easy to see that C encodes a 2-ranking of $K_{ac} \,\square\, K_{bd}$. Since the entries in C belong to $\{0, \ldots, k\ell - 1\}$, we have that $\chi_2(K_{ac} \,\square\, K_{bd}) \le k\ell$. $\square$

Proposition 4 may be iterated to obtain upper bounds on $\chi_2(K_m \,\square\, K_n)$.

Corollary 5. *If m and n are powers of 2 with $m \le n$, then $\chi_2(K_m \,\square\, K_n) \le nm^{\log_2(3)-1} \approx nm^{0.585}$.*

Proof. Observe that $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ is a 2-ranking witnessing that $\chi_2(K_2 \,\square\, K_2) \le 3$. If $m = 1$, then $\chi_2(K_m \,\square\, K_n) = n$, and so the bound holds. Otherwise, by Proposition 4 and induction, we have that $\chi_2(K_m \,\square\, K_n) \le \chi_2(K_{m/2} \,\square\, K_{n/2}) \cdot \chi_2(K_2 \,\square\, K_2) \le \frac{n}{2} \left(\frac{m}{2}\right)^{\log_2(3)-1} \cdot 3 = nm^{\log_2(3)-1}$. $\square$

When m and n are not powers of two, we may apply Corollary 5 to $K_{m'} \,\square\, K_{n'}$ where m' and n' are the least powers of two larger than m and n, respectively. Since $m' < 2m$ and $n' < 2n$, this gives $\chi_2(K_m \,\square\, K_n) < 3nm^{\log_2(3)-1}$ for general m and n. To prove a lower bound on $\chi_2(K_m \,\square\, K_n)$, we restrict the number of times that certain ranks can appear.

Lemma 6. *In a 2-ranking of $K_m \,\square\, K_n$, each column of height m contains k ranks which are assigned to at most k vertices for $1 \le k \le m$.*

Proof. Let A be an $(m \times n)$-matrix representing a 2-ranking of $K_m \,\square\, K_n$, and let x be the jth column in A. Let R be the set of rows containing the k highest ranks in x, and let $S = \{A(i,j) \colon i \in R\}$. We claim that each rank in S appears only in rows in R. Since each rank appears at most once in each row, it then follows that each of the k ranks in S is assigned to at most k vertices.

Suppose that $A(i,j) = A(i',j')$ where $i \in R$. Since A is a 2-ranking, it must be that $A(i',j) > A(i,j)$, which implies that $A(i',j)$ is among the k highest ranks in x. Therefore $i' \in R$ also. It follows that each rank in S appears only in rows in R. $\square$

Lemma 6 forces a nontrivial number of ranks in a 2-ranking of $K_m \,\square\, K_n$.

Theorem 7. *We have $\chi_2(K_m \,\square\, K_n) \geq nH_m$, where H_m is the harmonic sum $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}$.*

Proof. Let A be an $(m \times n)$-matrix encoding an optimal 2-ranking of $K_m \,\square\, K_n$, and let a_k be the number of ranks that A assigns to exactly k vertices. Note that $\chi_2(K_m \,\square\, K_n) = \sum_{k=1}^{m} a_k$. We claim that for $1 \leq k \leq m$, we have that $\sum_{i=1}^{k} ia_i \geq kn$. Indeed, $\sum_{i=1}^{k} ia_i$ counts the number of vertices in $K_m \,\square\, K_n$ whose ranks appear at most k times in A. By Lemma 6, for $1 \leq k \leq m$, each of the n columns in A is associated with k such vertices. Therefore $\sum_{i=1}^{k} ia_i \geq kn$ as claimed.

Let $a_1, \ldots, a_m$ minimize $\sum_{i=1}^{m} a_i$ subject to the conditions $\sum_{i=1}^{k} ia_i \geq kn$ for $k \in [m]$. We claim that in each constraint, equality holds. Indeed, if k is the least integer such that $\sum_{i=1}^{k} ia_i > kn$, then we may reduce a_k by a positive ε while still satisfying the constraints $\sum_{i=1}^{\ell} ia_i \geq \ell n$ for $1 \leq \ell \leq k$. If we also increase a_{k+1} by $\frac{k}{k+1}\varepsilon$, then all constraints are satisfied, but we have reduced $\sum_{i=1}^{m} a_i$ by $\frac{1}{k+1}\varepsilon$, contradicting the minimality of $\sum_{i=1}^{m} a_i$.

Since equality holds in all constraints, we conclude $a_k = n/k$ for each k and $\sum_{i=1}^{m} a_i = nH_m$. ☐

When $n \geq m!$, Theorem 7 gives the correct order of growth of $\chi_2(K_m \,\square\, K_n)$. In fact, equality holds when $m! \mid n$.

Theorem 8. *If $m! \mid n$, then $\chi_2(K_m \,\square\, K_n) = nH_m$.*

Proof. Theorem 7 gives the lower bound. We claim that it suffices to prove $\chi_2(K_m \,\square\, K_{m!}) \leq (m!)H_m$. Indeed, with $n = tm!$, the general case would then follow from Proposition 4, since $\chi_2(K_m \,\square\, K_n) \leq \chi_2(K_m \,\square\, K_{m!}) \cdot \chi_2(K_1 \,\square\, K_t) = (m!)H_m \cdot t = nH_m$.

We prove that $\chi_2(K_m \,\square\, K_{m!}) = (m!)H_m$ by induction on m. For $m = 1$, the statement is trivial. Suppose that $m \geq 2$ and let A' be an $((m-1) \times (m-1)!)$-matrix encoding an optimal 2-ranking of $K_{m-1} \,\square\, K_{(m-1)!}$. By shifting the ranks appropriately, let $A'_1, \ldots, A'_m$ be copies of A' that use disjoint intervals of ranks, starting with rank $(m-1)! + 1$. The ranks appearing in $A'_1, \ldots, A'_m$ are *high*, and the ranks in $[(m-1)!]$ are *low*.

We construct an $(m \times m!)$-matrix A encoding a 2-ranking of $K_m \,\square\, K_{m!}$ as follows. Let M_i be an $(m \times (m-1)!)$-matrix such that deleting the ith row of M_i gives A'_i and whose ith row contains each low rank. Let $A = [M_1 \cdots M_m]$. The rows and columns of A have distinct entries. Suppose that $A(i,j) = A(i',j')$. If $A(i,j)$ and $A(i',j')$ are both low ranks, then columns j and j' belong to distinct blocks of A and so $A(i',j)$ and $A(i,j')$ are both high ranks. If $A(i,j)$ and $A(i',j')$ are both high ranks, then columns j and j' belong to the same block of A and so the opposite corners have higher rank by induction. It follows that A is a 2-ranking. Since A uses $(m-1)!$ low ranks and $m \cdot [(m-1)!H_{m-1}]$ high ranks, we have that $\chi_2(K_m \,\square\, K_{m!}) \leq (m-1)! + m!H_{m-1} = m!(1/m + H_{m-1}) = m!H_m$. ☐

Using that $H_m = (1 + o(1)) \ln m$, we obtain an asymptotic formula for $\chi_2(K_m \,\square\, K_n)$ when m is constant.

Corollary 9. *For each positive integer m, we have that $\chi_2(K_m \,\square\, K_n) = (1+o(1))n \ln m$ as $n \to \infty$.*

Proof. The lower bound follows immediately from Theorem 7. For the upper bound, let n' be the least multiple of $m!$ that is at least n. By Theorem 8, we have $\chi_2(K_m \,\square\, K_n) \leq \chi_2(K_m \,\square\, K_{n'}) = n'H_m \leq (n + m!)H_m = (1 + (m!)/n) \cdot nH_m = (1 + o(1))n \ln m$. ☐

In the diagonal case, our bounds are far apart. Combining Theorem 7 and Corollary 5 gives $\Omega(n \log n) \leq \chi_2(K_n \,\square\, K_n) \leq O(n^{\log_2 3})$. What is the order of growth of $\chi_2(K_n \,\square\, K_n)$?

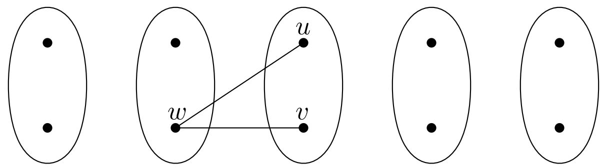

Figure 1: A forbidden configuration in a 2-ranking

4 The 2-ranking number of graphs with maximum degree k

Let G be a graph with $\Delta(G) = k$, where $\Delta(G)$ is the maximum degree of G. Since $\chi_s(G) \le \chi_2(G) \le \chi(G^2) \le \Delta(G^2) + 1 \le k^2 + 1$, it is interesting to ask for the maximum of $\chi_s(G)$ and $\chi_2(G)$ over all graphs G with maximum degree at most k. Fertin, Raspaud, Reed [4] proved that the maximum of $\chi_s(G)$ over all graphs with maximum degree at most k is at least $\Omega(\frac{k^{3/2}}{(\log k)^{1/2}})$ and is at most $O(k^{3/2})$. We make slight modifications to their probabilistic construction to show that the maximum of $\chi_2(G)$ over all graphs with maximum degree k is at least $\Omega(k^2/\log k)$. For each integer n, let $[n] = \{1, \ldots, n\}$.

Theorem 10. *For each k, there exists a graph G with $\Delta(G) \le k$ and $\chi_2(G) \ge \Omega(k^2/\log k)$.*

Proof. Choose n so that n is even and $2np \le k$, where $p = c(\log n/n)^{1/2}$ for some constant c to be chosen later. Since we may assume that k is sufficiently large, we may assume that n is also sufficiently large. Let G be a random graph chosen from $G(n,p)$. Each vertex in G has expected degree $(n-1)p$, and it is well known (see, for example, [2]) that $\mathbb{P}(\Delta(G) \le 2np) \to 1$ as $n \to \infty$. For each function $f\colon V(G) \to [n/2]$, let A_f be the bad event that f is a 2-ranking of G. Applying the union bound, we have that $\mathbb{P}(\chi_2(G) \le \frac{n}{2}) = \mathbb{P}(\bigcup_f A_f) \le \sum_f \mathbb{P}(A_f)$.

Fix a function $f\colon V(G) \to [n/2]$. Discarding one vertex from each rank class with an odd number of vertices, we may partition the remaining vertices into pairs $S_1, \ldots, S_\ell$ such that both vertices on S_i have the same rank under f. Since at most $n/2$ vertices are discarded, we have $\ell \ge (1/2)(n - n/2) = n/4$. Index the pairs so that $i \le j$ implies that $f(u) \le f(v)$ when $u \in S_i$ and $v \in S_j$. For each pair $\{S_i, S_j\}$ with $i < j$, the probability that G contains some path uwv such that $u, v \in S_j$ and $w \in S_i$ is at least p^2. If this happens, then f is not a 2-ranking since either $f(u) = f(v) = f(w)$ or $f(u) = f(v) > f(w)$; see Figure 1. Since the paths uwv form an edge-disjoint family as we range over the pairs $\{S_i, S_j\}$, it follows that the pairs $\{S_i, S_j\}$ give independent chances for A_f to fail. It follows that $\mathbb{P}(A_f) \le (1-p^2)^{\binom{n/4}{2}}$. For sufficiently large n, it follows that

$$\mathbb{P}(\chi_2(G) \le \frac{n}{2}) \le \sum_f \mathbb{P}(A_f) \le (n/2)^n (1-p^2)^{\binom{n/4}{2}} \le (n/2)^n e^{-\frac{p^2 n^2}{33}} = \left(\frac{n}{2n^{c^2/33}}\right)^n.$$

With $c = 6$, we have that $\mathbb{P}(\chi_2(G) \le \frac{n}{2}) \to 0$ as $n \to \infty$. It follows that with probability tending to 1, we have that $\Delta(G) \le k$ and $\chi_2(G) > n/2 \ge c' \frac{k^2}{\ln k}$ for some positive constant c'. □

5 The 2-ranking number of subcubic graphs

A graph G is *subcubic* if $\Delta(G) \le 3$. The *star list chromatic number* of G, denoted $\chi_s^\ell(G)$, is the minimum integer t such that if each vertex v in G is assigned a list $L(v)$ of t colors, there is a star coloring of G in which each vertex v receives a color from its list $L(v)$. Albertson, Chappell, Kierstead, Kündgen, and Ramamurthi [1] gave an elegant proof that every subcubic graph G satisfies $\chi_s^\ell(G) \le 7$. It follows that $\chi_s(G) \le \chi_s^\ell(G) \le 7$ when G is subcubic. Chen, Raspaud, and Wang [3] proved that every subcubic graph G satisfies $\chi_s(G) \le 6$.

Let G be the 3-regular graph obtained from C_8 by joining vertices at distance 4. Fertin, Raspaud, and Reed [4] proved that $\chi_s(G) = 6$, and it follows that the result of Chen, Raspaud, and Wang is best possible.

Here, we show that $\chi_2(G) \le 7$ when G is subcubic. Since $\chi_2(G) \ge \chi_s(G)$ always, the example of Fertin, Raspaud, and Reed shows that our bound cannot be reduced by more than 1 in the general case. Nonetheless, we believe the bound can be improved by 2 aside from a single exception; see Conjecture 12.

An *independent set* in G is a set of vertices that are pairwise nonadjacent. We use $N_G(u)$ for the set of neighbors of u in G and $N_G[u]$ for the *closed neighborhood* $N_G(u) \cup \{u\}$. When $S \subseteq V(G)$, we use $G[S]$ for the subgraph of G induced by S. Vertices u and v in a graph G are *antipodal* if $\mathrm{dist}(u, v) = \mathrm{diam}(G)$, where $\mathrm{diam}(G)$ is the maximum distance between a pair of vertices in G.

Theorem 11. *If G is subcubic, then $\chi_2(G) \le 7$.*

Proof. Let G be a subcubic graph. We may assume that G is connected. Let S be a maximal independent subset of $V(G)$, and let $\overline{S} = V(G) - S$. Since S is maximal, every vertex in G is in S or has a neighbor in S. We claim that in G^2, each vertex $v \in \overline{S}$ has at most 6 neighbors in $\overline{S}$. Indeed, for each $u \in N(v)$, let $A_u = N_G[u] - v$. Note that $|A_u \cap \overline{S}| \le 2$, or else $N_G[u] \subseteq \overline{S}$, contradicting the choice of S. Since v has at most 3 neighbors, the claim follows.

Therefore $\Delta(G^2[\overline{S}]) \le 6$. If $G^2[\overline{S}]$ does not contain a copy of K_7, then by Brooks's theorem, $\chi(G^2[\overline{S}]) \le 6$. Using a proper coloring of $G^2[\overline{S}]$ with colors in $[6]$ and assigning rank 0 to all vertices in S gives a 2-ranking of G. Indeed, paths of length 1 are well-ranked and G contains no paths of length 2 joining vertices with nonzero ranks.

Hence we may assume that $G^2[\overline{S}]$ contains a copy of K_7. Since G is connected, it follows that $G^2[\overline{S}]$ is connected. Since $G^2[\overline{S}]$ is connected and has maximum degree at most 6, we have $G^2[\overline{S}] = K_7$.

This has several implications for the structure of $G[\overline{S}]$. First, we claim that every vertex in $G[\overline{S}]$ has degree 0 or degree 2. Since each vertex $u \in \overline{S}$ has a neighbor in S, it follows that u has at most 2 neighbors in $G[\overline{S}]$. If the only neighbor of u in $G[\overline{S}]$ is v, then v has at most 5 neighbors in $G^2[\overline{S}]$: at most 2 from each of the neighbors of v in G besides u, and u itself. This contradicts that $G^2[\overline{S}] = K_7$.

It follows that $G[\overline{S}]$ is a 7-vertex graph whose components are isolated vertices and cycles. We claim that each cycle in $G[\overline{S}]$ has length at least 5. Indeed, suppose that v is in a cycle C in $G^2[\overline{S}]$ of length at most 4, and let u_1 and u_2 be the neighbors of v along C. Each neighbor of v in G contributes at most 2 neighbors of v in $G^2[\overline{S}]$. Since C has length at most 4, the contributions of u_1 and u_2 have nonempty intersection. It follows that v has fewer than 6 neighbors in $G^2[\overline{S}]$, contradicting that $G^2[\overline{S}] = K_7$.

Suppose that $G[\overline{S}]$ contains a 5-cycle C, and let x and y be the vertices in $G[\overline{S}]$ that are not in C. Let u be a vertex in C. Since u is adjacent to x and y in $G^2[\overline{S}]$, it must be that u is adjacent in G to a vertex $s_u \in S$ whose other two neighbors in G are x and y. The vertices $\{s_u \colon u \in V(C)\}$ have distinct neighborhoods of size 3 and are therefore distinct. This is not possible, since x and y would have 5 neighbors in G. Therefore $G[\overline{S}]$ does not contain a 5-cycle.

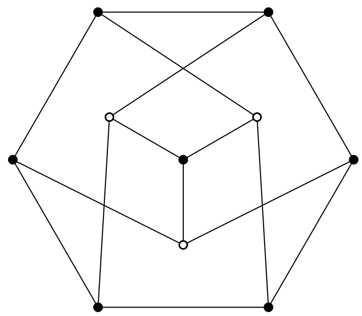

Figure 2: Vertices in S are white and vertices in $\overline{S}$ are black.

Suppose that $G[\overline{S}]$ contains a 7-cycle C, and let u be a vertex in C. Since u is adjacent in $G^2[\overline{S}]$ to the two vertices x and y at distance 3 from u in C, it must be that u is adjacent in G to a vertex s_u such that $N_G(s_u) = \{u, x, y\}$. Again, the vertices $\{s_u \colon u \in V(C)\}$ have distinct neighborhoods of size 3, and are therefore distinct. This is impossible, since x is adjacent in G to s_u, s_x, and its two neighbors on C. Therefore, $G[\overline{S}]$ cannot contain a 7-cycle.

It follows that either $G[\overline{S}] = C_6 + K_1$ or $G[\overline{S}] = 7K_1$. Suppose that $G[\overline{S}]$ contains a 6-cycle C and let x be the isolated vertex. If u is a vertex on C, then u is adjacent in G to a vertex s_u whose neighbors are u, x, and the vertex on C antipodal to u. It follows that G is the Petersen graph, as in Figure 2. Suppose that $G[\overline{S}] = 7K_1$. It follows that each vertex u in $\overline{S}$ is adjacent in G to 3 neighbors v_1, v_2, v_3 in S. Moreover, we have $\bigcup_{i=1}^{3} N_G(v_i) = \overline{S}$ and $N_G(v_i) \cap N_G(v_j) = \{u\}$ for $i \neq j$. Since G is connected, we have that G is a 3-regular $(S, \overline{S})$-bigraph, and so $|S| = |\overline{S}| = 7$. It follows also that G does not contain a copy of C_4, or else some vertex $u \in \overline{S}$ would have neighbors v_1 and v_2 with $|N_G(v_1) \cap N_G(v_2)| \geq 2$. Since G is a bipartite 3-regular graph on 14 vertices with girth at least 6, it follows that G is the Heawood graph.

As we have seen, if G is subcubic, then $\chi_2(G) \leq 7$, or G is the Petersen graph, or G is the Heawood graph. If G is the Petersen graph, then G contains a maximal independent set S of size 4. We may repeat the argument with $G[\overline{S}]$ having 6 vertices. If G is the Heawood graph, then a vertex u and the four vertices antipodal to u form a maximal independent set S of size 5. We we may repeat the argument with $G[\overline{S}]$ having 9 vertices. □

Besides the example of Fertin, Raspaud, and Reed, we are not aware of another subcubic graph that requires 6 ranks for a 2-ranking. Plausible candidates such as the Petersen graph and the Heawood graph admit 2-rankings with only 5 ranks.

Conjecture 12. *If G is subcubic, then $\chi_2(G) \leq 6$ and equality holds if and only if G is the cubic graph obtained from C_8 by joining vertices at distance 4.*

6 The product of a triangle and a cycle

Applied to the product of a pair of cycles, Corollary 3 states that $\chi_2(C_m \,\square\, C_n) = 5$ when m and n are divisible by 4. In this section, we show that the 2-ranking number of cycle products may depend on the parity of the lengths of the factors. In particular, we show that for sufficiently large n, the 2-ranking number of $C_3 \,\square\, C_n$ is 5 when n is even and 6 when n is odd. We represent a 2-ranking of $C_3 \,\square\, C_n$ with a $(3 \times n)$-array A such that $A(i, j)$ is the rank of $(u_i, v_j) \in V(C_3 \,\square\, C_n)$.

Lemma 13. *If $n \geq 24$, then $\chi_2(C_3 \,\square\, C_n) \leq 6$.*

Proof. Let $n = 4q + r$ for integers q and r with $r \in \{0, 1, 2, 3\}$. Since $q \geq 6$, we have that $n = 4(q - 2r) + 9r$, and so n is a nonnegative integer combination of 4 and 9. We give 2-rankings of $C_3 \,\square\, C_9$ and $C_3 \,\square\, C_4$ that can be appended together to give a 2-ranking of $C_3 \,\square\, C_n$.

2	4	0	3	1	0	4	0	5
0	5	1	0	5	2	0	1	3
1	3	2	4	0	3	5	2	4

2	4	0	5
0	5	1	3
1	3	2	4

To see that these are 2-rankings, observe that the vertices assigned rank 0 form an independent set, and for each positive rank t, the vertices assigned rank t are independent in $(C_3 \,\square\, C_n)^2$. Because both 2-rankings agree on the first two columns and the last two columns, appending the arrays gives a 2-ranking. □

Our upper bound improves for even n.

Lemma 14. *If n is even and $n \geq 4$, then $\chi_2(C_3 \,\square\, C_n) \leq 5$.*

Proof. Let $n = 4q + 6r$ for integers q and r with $r \in \{0, 1\}$. We give 2-rankings of $C_3 \,\square\, C_4$ and $C_3 \,\square\, C_6$ which can be appended to give a 2-ranking of $C_3 \,\square\, C_n$.

0	1	0	2
3	2	4	1
4	0	3	0

0	1	3	0	4	2
3	0	4	2	0	1
4	2	0	1	3	0

Regardless of how these arrays are appended, vertices assigned rank 0 form an independent set, and for each positive rank t, the vertices assigned rank t form an independent set in $(C_3 \,\square\, C_n)^2$. □

Since $C_3 \,\square\, C_n$ has degeneracy 4, it follows that $\chi_2(C_3 \,\square\, C_n) \geq 5$ always. When n is odd, our lower bound improves.

Lemma 15. *If n is odd, then $\chi_2(C_3 \,\square\, C_n) > 5$.*

Proof. Suppose for a contradiction that $C_3 \,\square\, C_n$ has a 2-ranking A using ranks in $[5]$. Ranks 4 and 5 are *high*; the other ranks are *low*. Note that each high rank appears at most once in every pair of adjacent columns of A. It follows that at most k vertices are assigned to each high rank. A column containing all of the low ranks is *low*, and a column containing all of the high ranks is *high*. Since A has $2k + 1$ columns and at most $2k$ vertices have high rank, it follows that some column of A is low.

It is easy to check that $\chi_2(C_3 \,\square\, P_2) \geq 5$. It follows that a column adjacent to a low column must be high. Since high ranks cannot appear in adjacent columns, a column adjacent to a high column must be low. Therefore the columns of A alternate high and low cyclically, contradicting that n is odd. □

Collecting the lemmas, we obtain the following theorem.

Theorem 16. *If n is odd and $n \geq 25$, then $\chi_2(C_3 \,\square\, C_n) = 6$. If n is even and $n \geq 6$, then $\chi_2(C_3 \,\square\, C_n) = 5$.*

It would be interesting to find the 2-ranking number of $C_m \,\square\, C_n$ for general m and n.

Acknowledgements

This research was supported in part by NSA grant H98230-14-1-0325.

References

[1] Michael O. Albertson, Glenn G. Chappell, H. A. Kierstead, André Kündgen, and Radhika Ramamurthi. Coloring with no 2-colored P_4's. *Electron. J. Combin.*, 11(1):Research Paper 26, 13, 2004.

[2] Béla Bollobás. *Random graphs.* Springer, 1998.

[3] Min Chen, André Raspaud, and Weifan Wang. 6-star-coloring of subcubic graphs. *Journal of Graph Theory*, 72(2):128–145, 2013.

[4] Guillaume Fertin, André Raspaud, and Bruce Reed. Star coloring of graphs. *Journal of Graph Theory*, 47(3):163–182, 2004.

[5] Joseph A. Gallian. A dynamic survey of graph labeling. *Electron. J. Combin.*, 5:Dynamic Survey 6, 43 pp. (electronic), 1998.

[6] Ilan Karpas, Ofer Neiman, and Shakhar Smorodinsky. On vertex rankings of graphs and its relatives. *Discrete Mathematics*, 338(8):1460–1467, 2015.

[7] Patric R. J. Östergård. On a hypercube coloring problem. *J. Combin. Theory Ser. A*, 108(2):199–204, 2004.

[8] Attila Pór and David R. Wood. Colourings of the Cartesian product of graphs and multiplicative Sidon sets. *Combinatorica*, 29(4):449–466, 2009.

[9] Peng-Jun Wan. Near-optimal conflict-free channel set assignments for an optical cluster-based hypercube network. *Journal of Combinatorial Optimization*, 1(2):179–186, 1997.