Parity Edge-Coloring of Graphs

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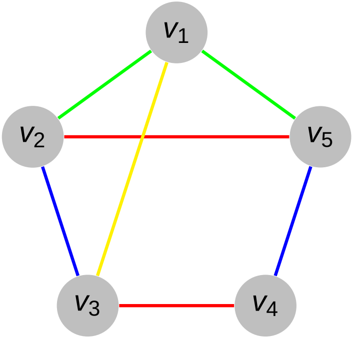
Parity Vectors



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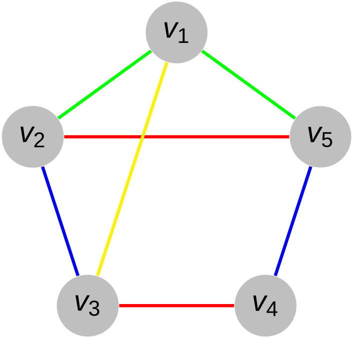

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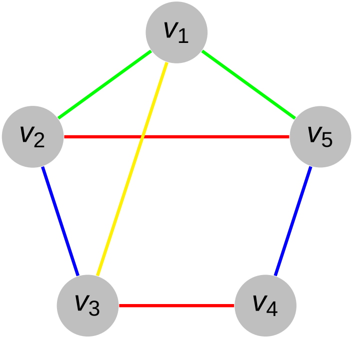

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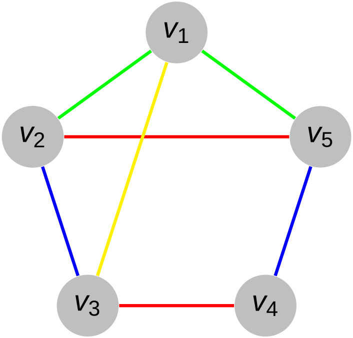
- We also abuse notation and use $\pi_f(W)$ as the set of colors that appear an odd number of times in W

Parity Vectors



Example

$W = v_1 v_2 v_5 v_1 v_3 v_2$

$$\pi(W) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

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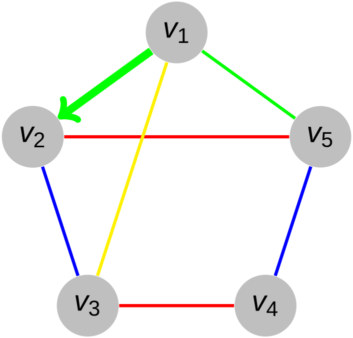

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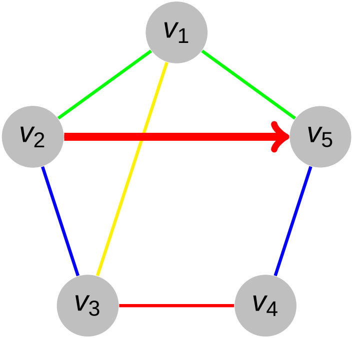

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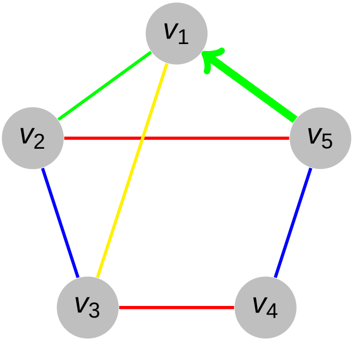

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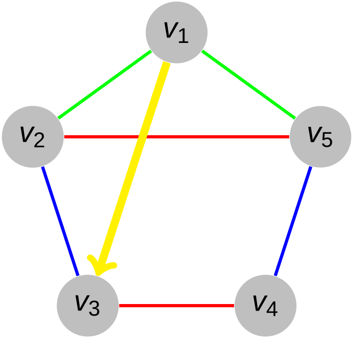

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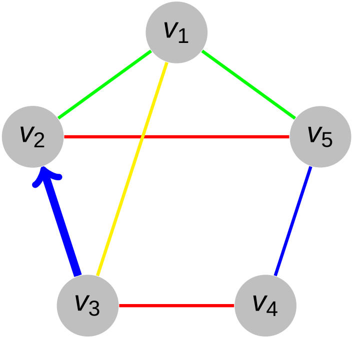

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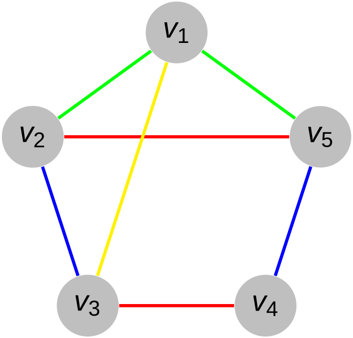

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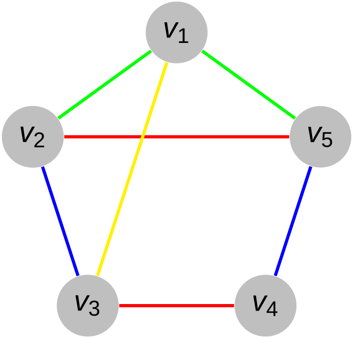

Definition

A parity walk is a walk W with $\pi(W) = \overrightarrow{0}$.

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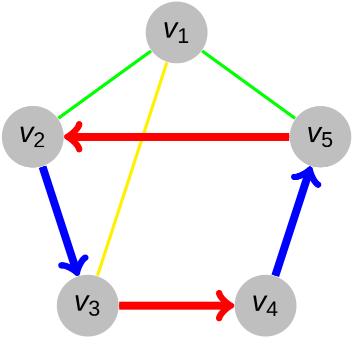

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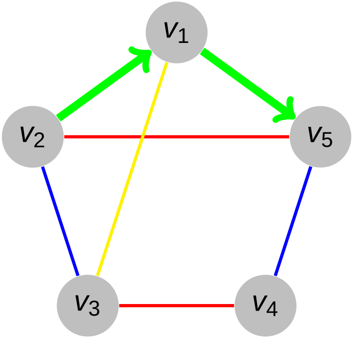

Definition

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- Parity walks can be closed ...
- ... or open.

Hypercubes and Parity Walks

Notation

If W_1 is a uv-walk and W_2 is a vw-walk, then W_1W_2 is the uw-walk given by the concatenation of W_1 and W_2. Similarly, $\overline{W_1}$ is the vu-walk obtained by reversing W_1.

Definition

The hypercube Q_k is the graph with vertex set $\{0, 1\}^k$ with an edge between u and v iff u and v differ in 1 coordinate.

Hypercubes and Parity Walks

Theorem (Havel, Movárek (1972))

Let G be a connected graph. G is a subgraph of Q_k iff there is an edge-coloring of G using at most k colors such that

$$\forall W \quad W \text{ is a parity walk} \iff W \text{ is closed}$$

Hypercubes and Parity Walks

Proof.

($\Longrightarrow$). Color an edge e in G according to the coordinate of Q_k that e crosses.

$\square$

Hypercubes and Parity Walks

Proof.

($\Longleftarrow$). Fix such an edge-coloring, let r be a vertex in G, let T be a spanning tree of G, and for each vertex u, let P_u be the ru-path in T. We define an embedding $\phi : V(G) \to V(Q_k)$ via

$$\phi(u) = \pi(P_u).$$

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- ϕ is injective: If $\phi(u) = \phi(v)$, then $\overline{P_u}P_v$ is a parity walk and hence closed, so $u = v$.

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- ϕ respects edges: Let $uv \in E(G)$. Then $\overline{P_u}P_vvu$ is closed and hence a parity walk. It follows that $\phi(u)$ and $\phi(v)$ differ only in the coordinate indexed by the color on uv.

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- All graphs have an edge-coloring in which every parity walk is closed

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A strong parity edge-coloring (spec) is an edge-coloring such that

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- In any edge-coloring of a tree, every closed walk is a parity walk.

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Corollary

A tree T is a subgraph of Q_k iff there is a spec of T using at most k colors.

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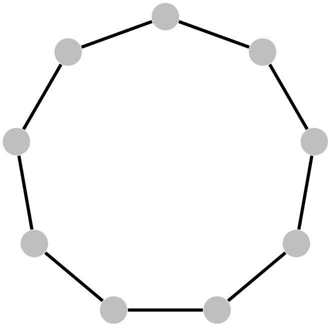
- Even cycles: $\widehat{p}(C_{2n}) = \lceil \lg 2n \rceil$
- Odd cycles: $\widehat{p}(C_{2n+1}) = ?$

What is $\widehat{p}(C_n)$ when n is odd?



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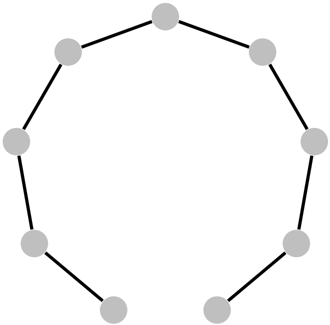

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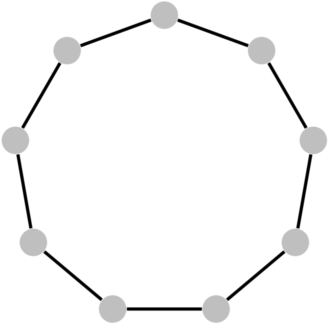

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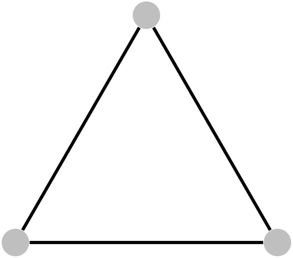

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Example

$$\widehat{p}(C_3) \in \{2, 3\}$$

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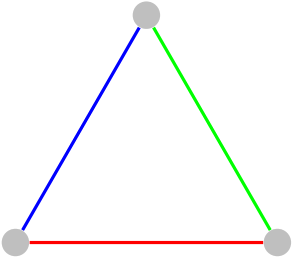

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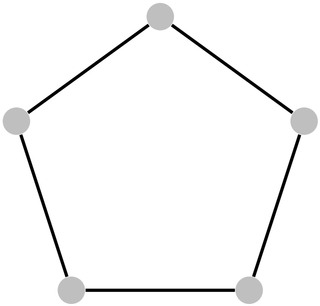

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Example

$$\widehat{p}(C_5) \in \{3, 4\}$$

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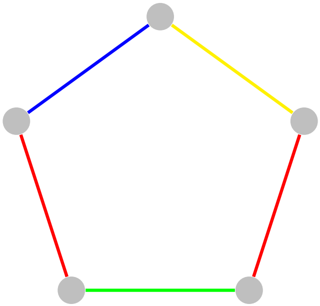

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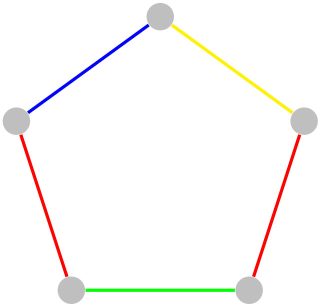

Theorem

For odd n,

$$\widehat{p}(C_n) = \lceil \lg n \rceil + 1.$$

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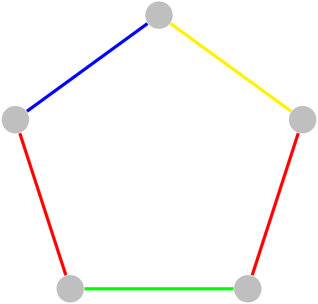

Proof.

We show $\widehat{p}(P_{2n}) \leq \widehat{p}(C_n)$.

- Fix a spec on C_n.

□

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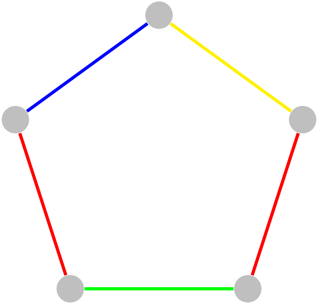

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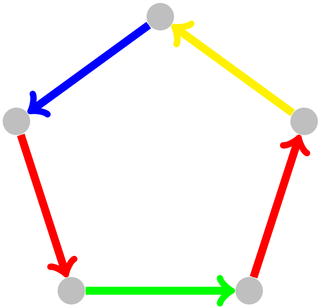

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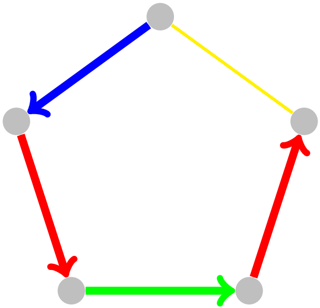

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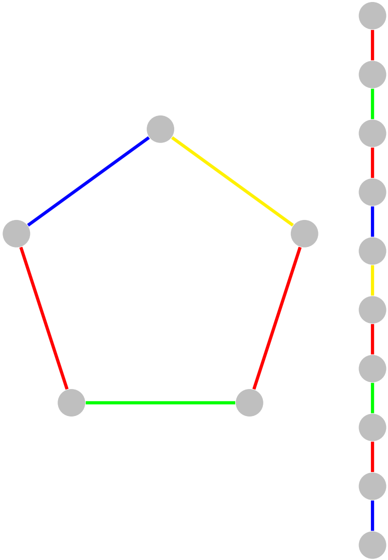

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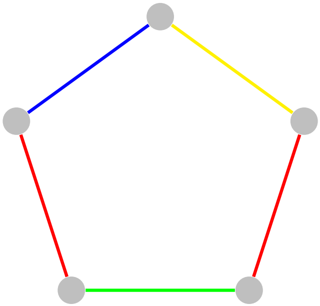

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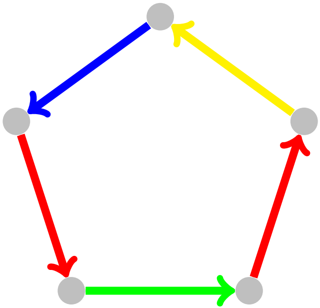

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- Open walks that lift to closed walks have odd length.

□

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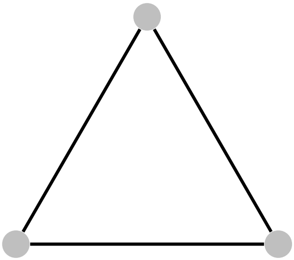

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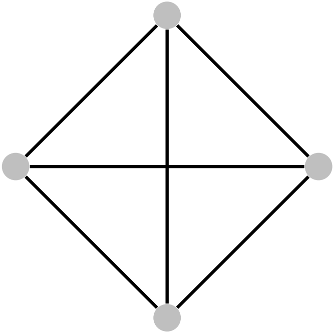

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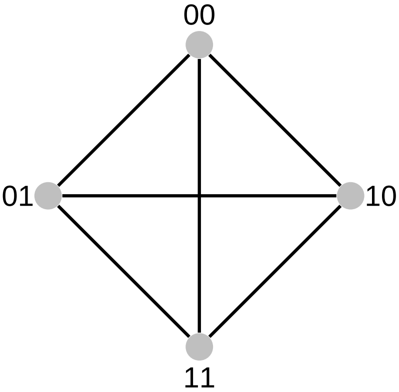

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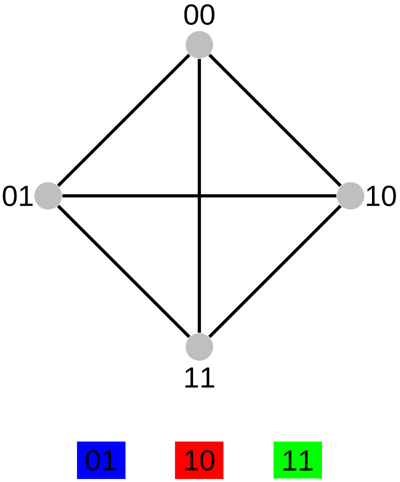

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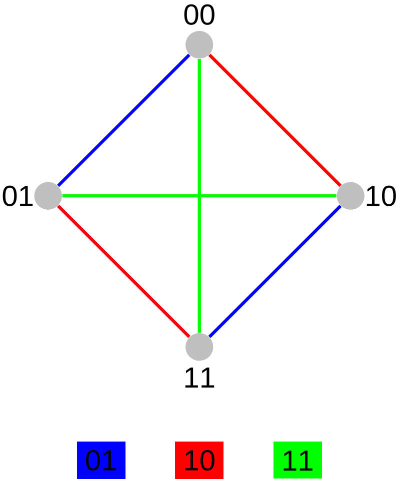

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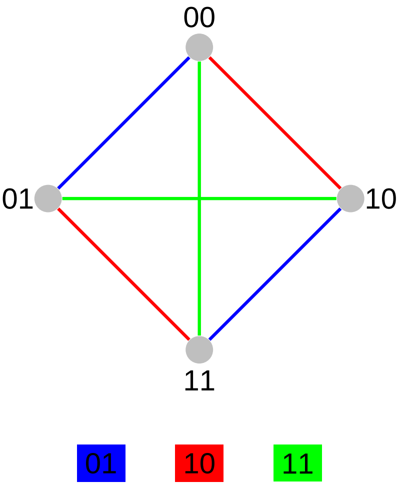

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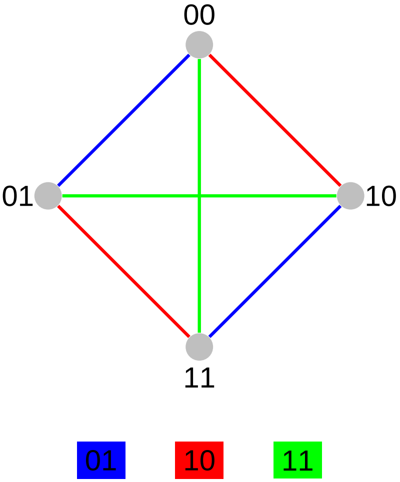

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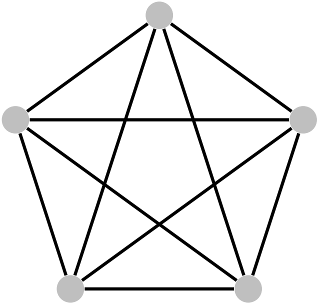

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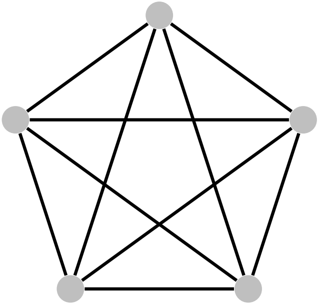

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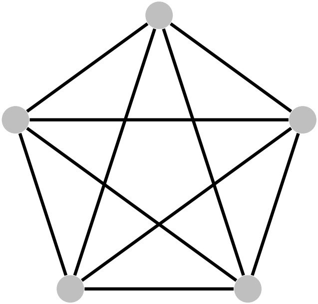

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If n is not a power of two, then $\widehat{p}(K_n) = \widehat{p}(K_{n+1})$.

- Strategy: add vertex, color new edges without introducing an open parity walk.
- We have a lot to worry about.

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Lemma (Spec Characterization)

Fix an edge-coloring of K_n. There is an open parity walk iff there is a closed walk W with $|\pi(W)| = 1$.

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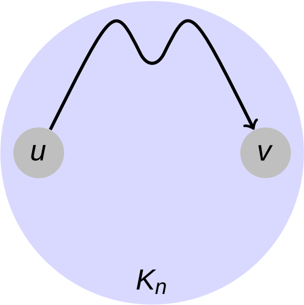

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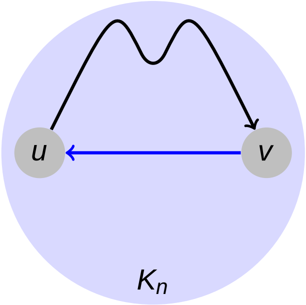

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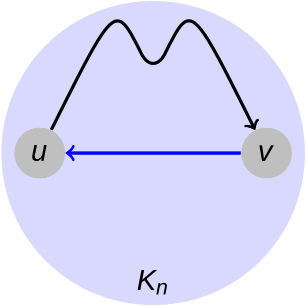

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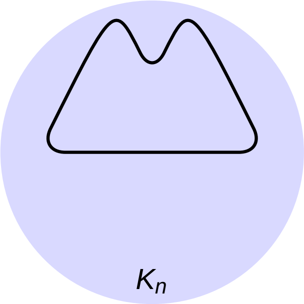

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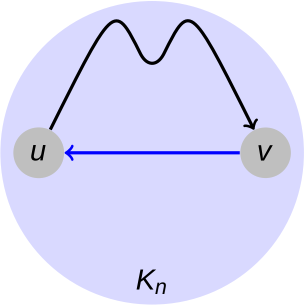

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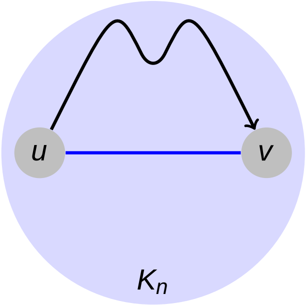

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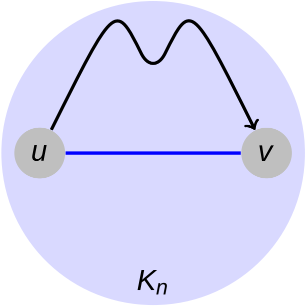

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- Let W' be the uv-walk obtained by removing vu
- W' is an open parity walk

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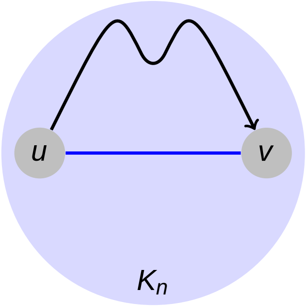

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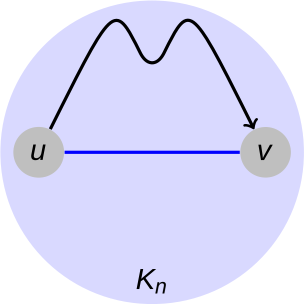

- Augmentation only worries about introducing closed walks W with $|\pi(W)| = 1$
- Linear algebra means we can worry even less!

The Parity Space

Proposition

Let f be an edge-coloring of a connected graph G. The parity space of f is

$$L_f = \{\pi(W) : W \text{ is closed}\}.$$

L_f is a linear subspace of $\mathbb{F}_2^k$.

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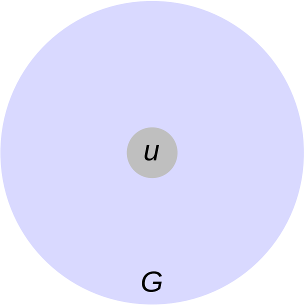

Proof.

- Let $W = u$. $\pi(W) = \overrightarrow{0} \in L_f$

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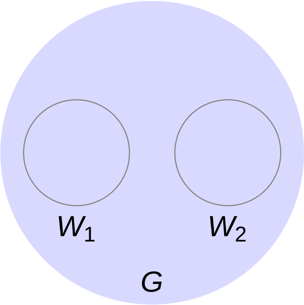

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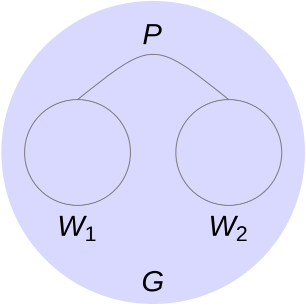

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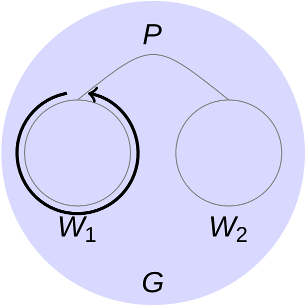

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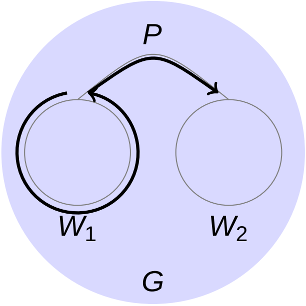

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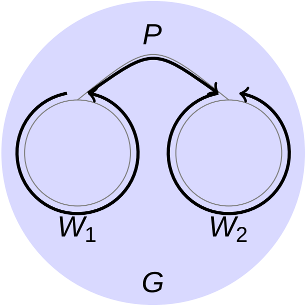

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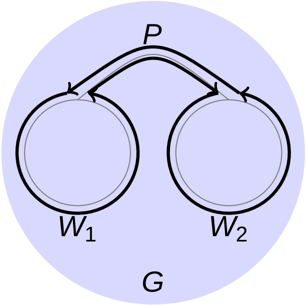

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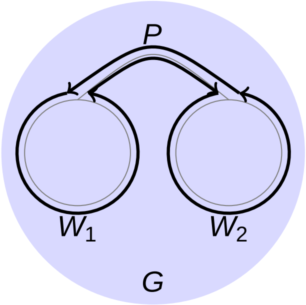

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- Let $W = W_1 P W_2 \overline{P}$
- $\pi(W) = \pi(W_1) + \pi(W_2) \in L_f$

□

A Parity Space Basis

Lemma (Parity Space Basis)

Let f be an edge-coloring of a graph G with a dominating vertex v. Then

$$\{\pi(T) : T \text{ is a triangle containing } v\}$$

is a basis for L_f.

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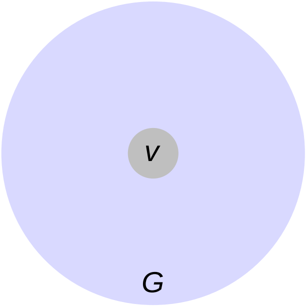

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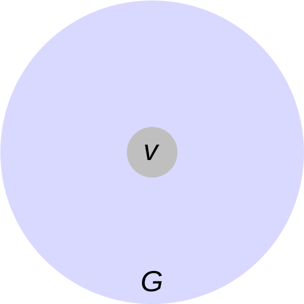

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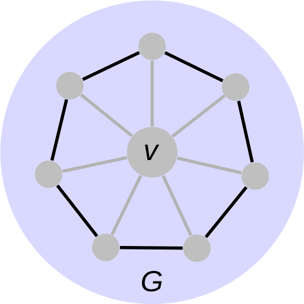

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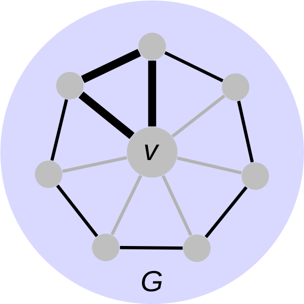

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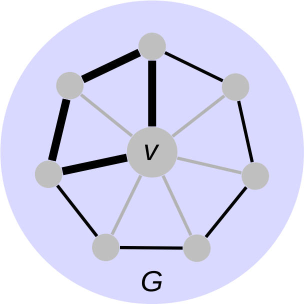

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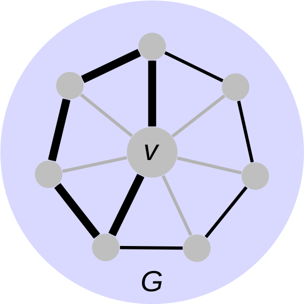

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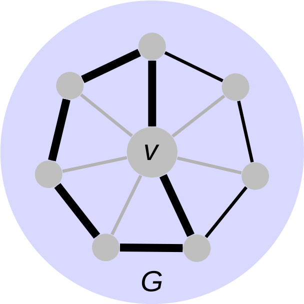

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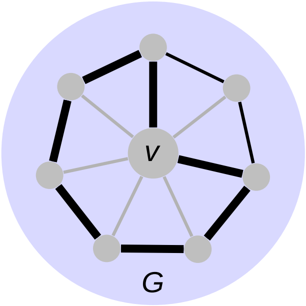

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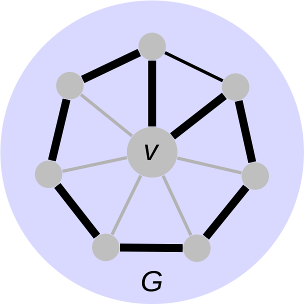

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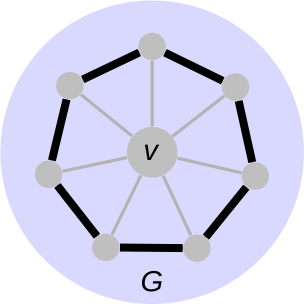

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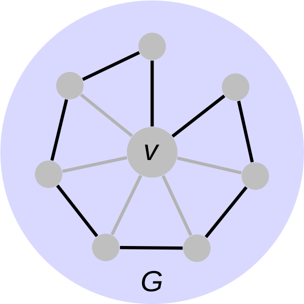

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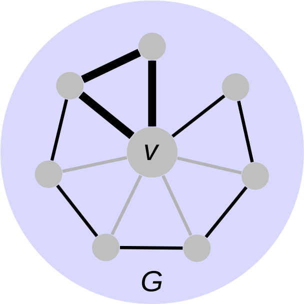

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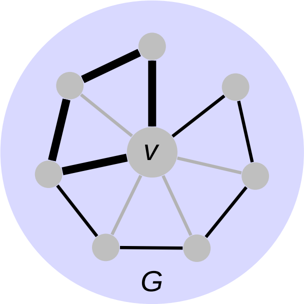

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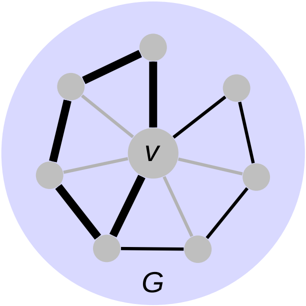

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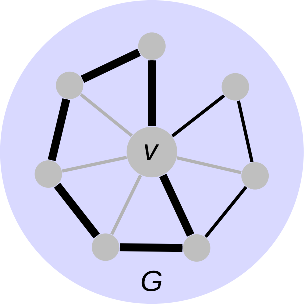

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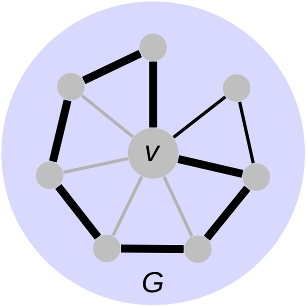

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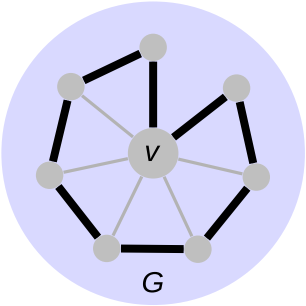

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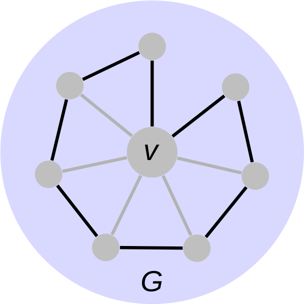

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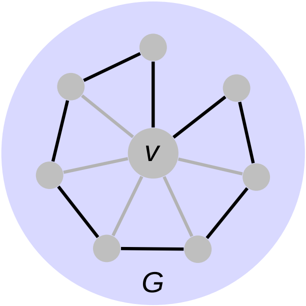

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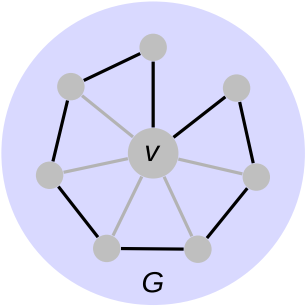

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- Argue K_n has a rich parity space, before augmentation

Triple Color Lemma

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Let f be a minimum spec of K_n. Then for every pair of colors $\{a, b\}$, there is a third color c and a closed walk W with $\pi(W) = \{a, b, c\}$.

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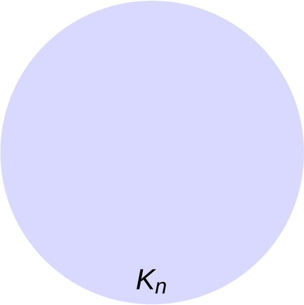

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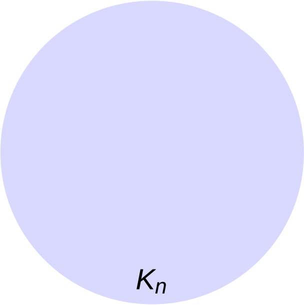

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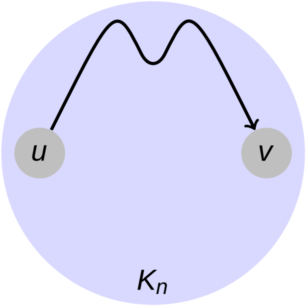

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- g is not a spec
- Let W' be a parity uv-walk

□

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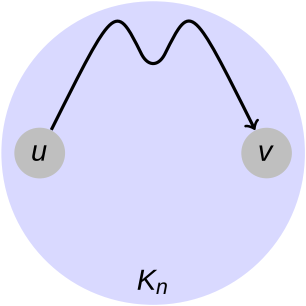

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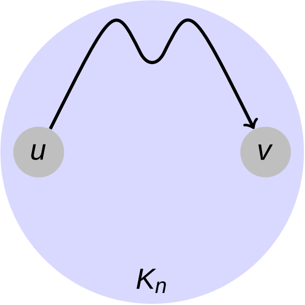

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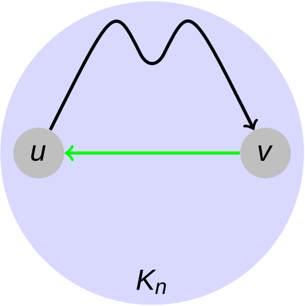

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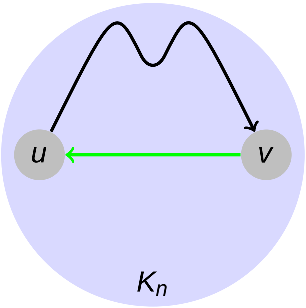

Proof.

- $\pi_g(W') = \emptyset$
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- Let $c = f(uv)$, let $W = W'vu$
- $c \notin \{a, b\}$

□

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Lemma (Augmentation)

If n is not a power of two, then $\widehat{p}(K_n) = \widehat{p}(K_{n+1})$.

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A spec of G is perfect if it uses $\Delta(G)$ colors. If f is a perfect spec of K_n, then n is a power of two and f is the canonical coloring.

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A spec of G is perfect *if it uses $\Delta(G)$ colors. If f is a perfect spec of K_n, then n is a power of two and f is the canonical coloring.*

Proof (sketch).

Starting with a single vertex, the proof finds larger and larger canonically colored subgraphs of K_n inductively. □

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- If n is not a power of two, each vertex misses a color

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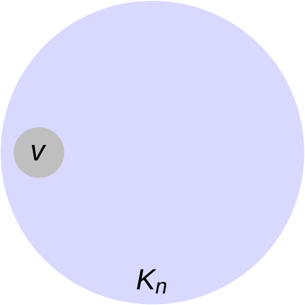

Proof.

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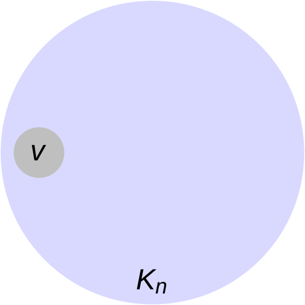

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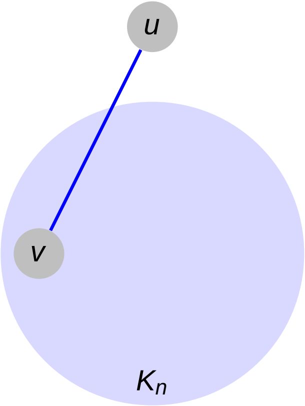

Proof.

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- Introduce a new vertex u. Color uv with a

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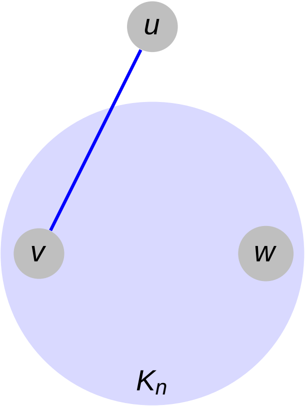

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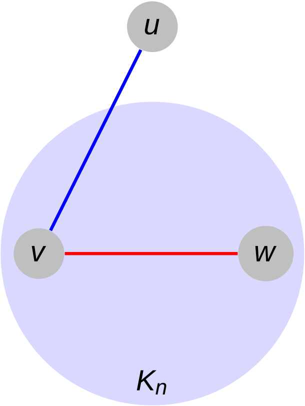

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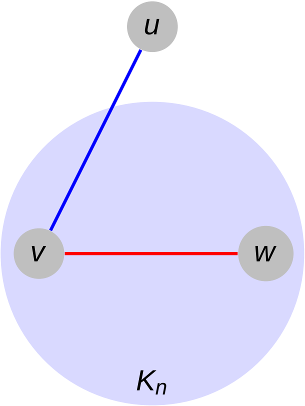

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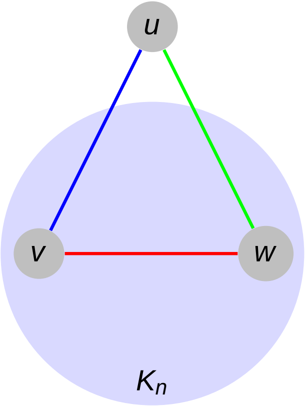

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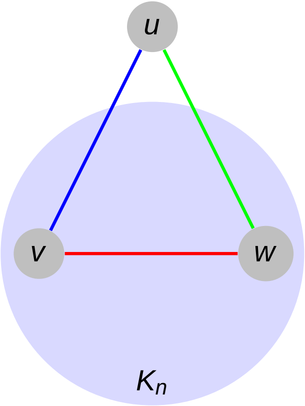

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- Let g be the coloring of K_{n+1}.

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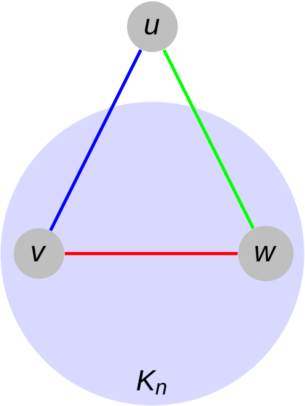

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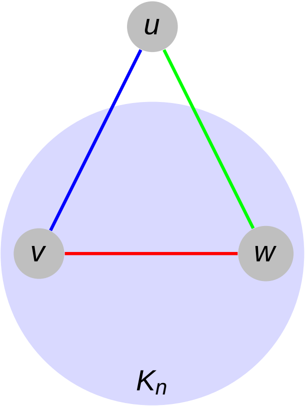

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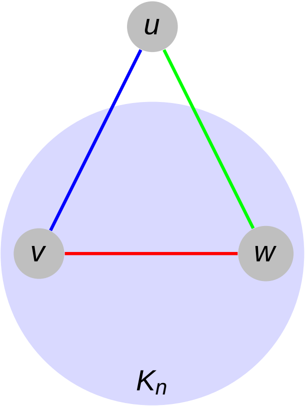

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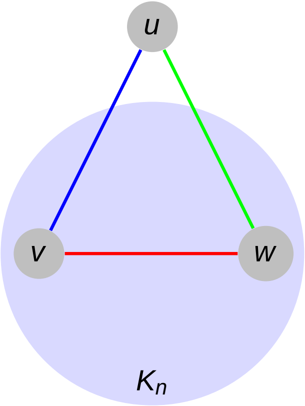

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- If $u \notin T$, then $\pi_g(T) = \pi_f(T) \in L_f$.

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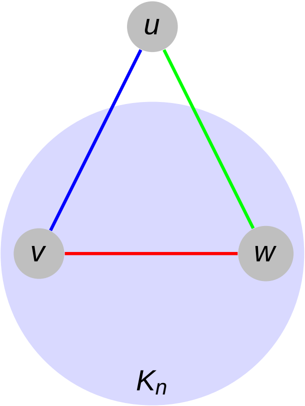

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- Otherwise, $T = uvwu$ for some w in K_n and $\pi_g(T) = \pi_f(W) \in L_f$ for some closed walk W by definition of g.

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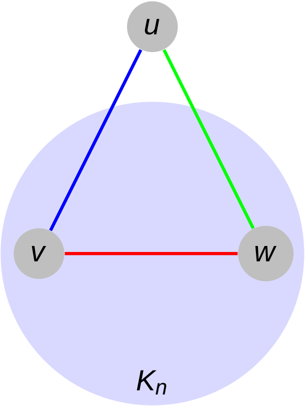

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- Otherwise, $T = uvwu$ for some w in K_n and $\pi_g(T) = \pi_f(W) \in L_f$ for some closed walk W by definition of g.
- Hence, g is a spec.

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An Application

Definition

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Example

Symmetric difference $f(x_1, x_2) = x_1 \triangle x_2$ is a nontrivial boolean function: $\mathcal{S} = \{\{1\}, \{2\}\}$.

An Application

Theorem (Daykin, Lovász (1974))

Let f be a nontrivial boolean function and let $\mathcal{F}$ be a family of n finite sets. Then

$$|\{f(A_1, \ldots, A_k) : \forall i \;\; A_i \in \mathcal{F}\}| \geq n.$$

An Application

Theorem (Daykin, Lovász (1974))

Let $\mathcal{F}$ be a family of n finite sets, and let

$$\mathcal{G} = \{A_1 \triangle A_2 : A_1 \neq A_2 \text{ and } A_1, A_2 \in \mathcal{F}\}.$$

Then $|\mathcal{G}| \geq n - 1$. If n is not a power of two, then $|\mathcal{G}| \geq n$.

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Quotation (*with changes in notation*)

"The example where $\mathcal{F}$ is all subsets of a [finite set] show that the theorem is best possible. Closer examination of the proof shows that if $|\mathcal{G}| = n - 1$ then $\mathcal{F}$ is very similar to the former example, but details are omitted."

An Application

Corollary

Let $\mathcal{F}$ be a family of n finite sets, and let

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Then $|\mathcal{G}| \geq 2^{\lceil \lg n \rceil} - 1$.

An Application

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Proof.

View $\mathcal{F}$ as the vertex set of K_n. Coloring an edge A_1A_2 with the symmetric difference of A_1 and A_2, we obtain a spec of K_n using only colors from $\mathcal{G}$. The bound on $|\mathcal{G}|$ follows. □

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- Is it $O(\log n)$?

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Theorem

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- Lower bounds apply to $|\{A_1 \triangle A_2 : A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2\}|$ with $m = |\mathcal{F}_1|$ and $n = |\mathcal{F}_2|$.

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- In general, $p(G) \neq \widehat{p}(G)$. Does equality hold for all bipartite graphs?

Stability of the Canonical Coloring

Question (Dhruv Mubayi)

Is there a (strong) parity edge-coloring of K_{2k} which uses only $(1 + o(1))2^k$ colors but is "far" from the canonical coloring?

Thanks

- Many other open problems in our paper.
- Thank You.