Online Degree Ramsey Theory

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University of Illinois at Urbana-Champaign

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Bloomington, IN
5 April 2008

Online Ramsey Theory

- Let's play a game:

Online Ramsey Theory

Picture of a Construction Worker

- Let's play a game:
  - Builder vs.

Online Ramsey Theory

Picture of a Painter

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Online Ramsey Theory

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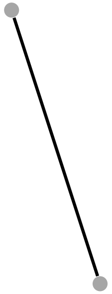

- Let's play a game:
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- Builder presents an edge

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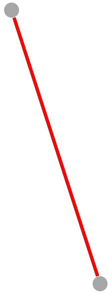

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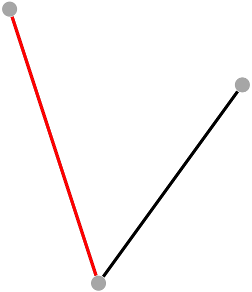

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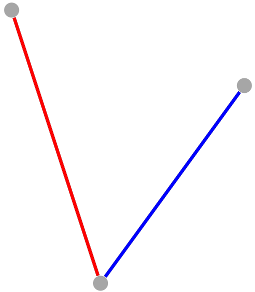

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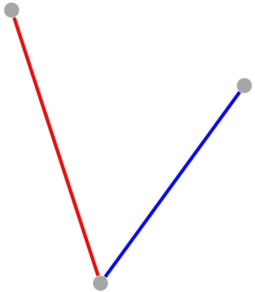

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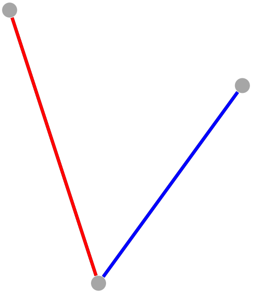

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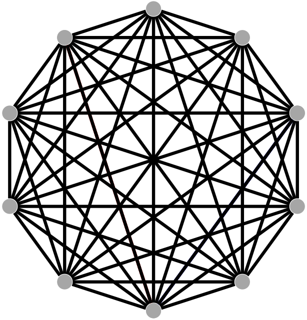

$\langle \text{Builder} \rangle$: Aha! Your move!

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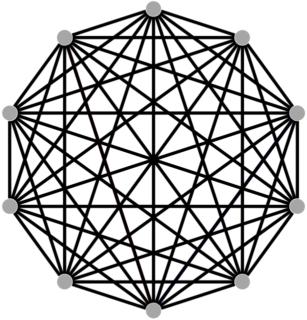

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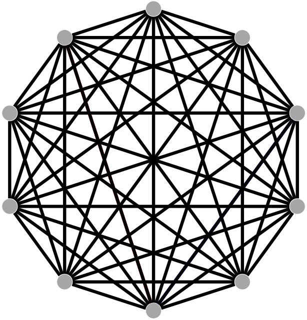

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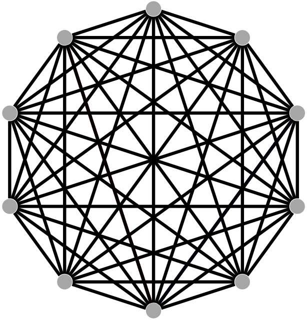

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- This defines the game $(G, \mathcal{H})$

Previous Results

Proposition (Grytczuk, Hałuszczak, Kierstead (2004))

If G is a forest, then Builder wins $(G, \mathrm{forests})$.

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Conjecture (GHK)

Builder wins $(G, \text{planar graphs})$ if and only if G is outerplanar.

Online Degree Ramsey Number

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- For each graph G, define the online degree-Ramsey number as follows:

$$\mathrm{odr}(G) = \min\{k : \text{Builder wins } (G, \mathcal{S}_k)\}$$

Warm Up

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- What is $\mathrm{odr}(K_{1,m})$?

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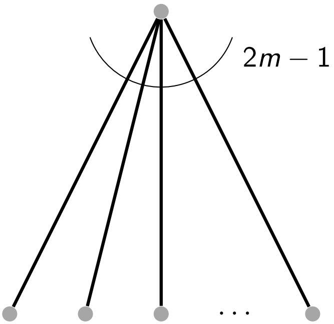

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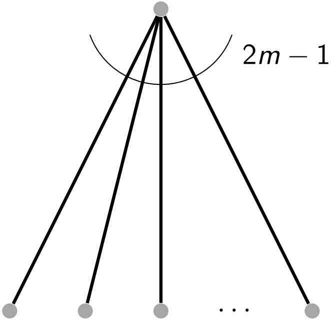

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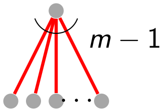

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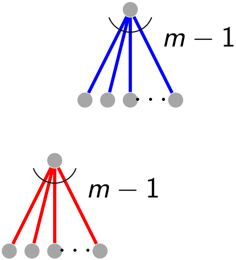

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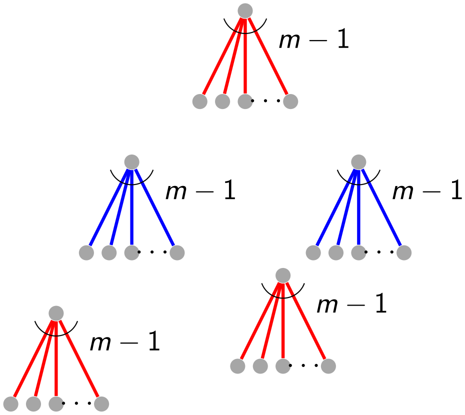

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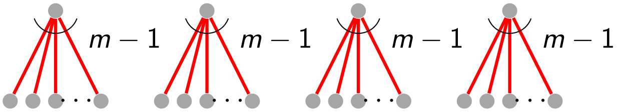

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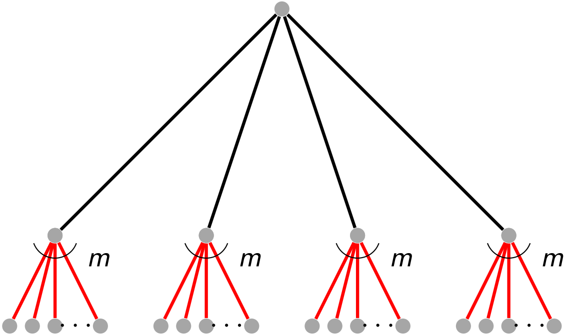

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- Present a new star

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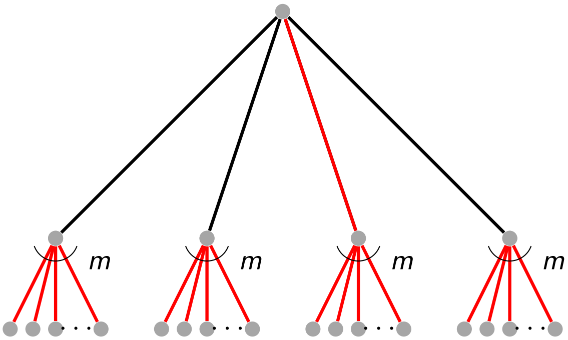

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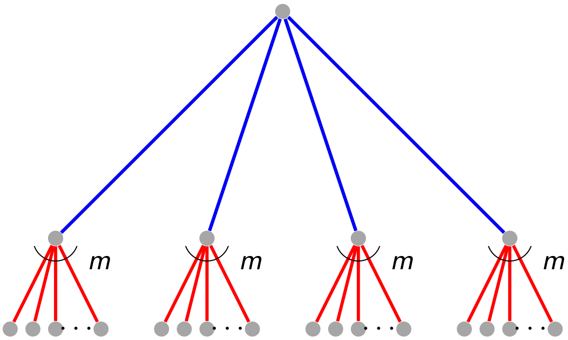

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- If any edge is red: we have a red $K_{1,m}$
- All edges blue: we have a blue $K_{1,m}$

The Greedy Painter

Finding strategies for Painter is difficult.

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- Whenever Painter colors an edge uv blue, either u or v is incident to at least $m - 1$ red edges.

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Theorem

If T is a tree, then $\mathrm{odr}(T) \leq 2\Delta(T) - 1$.

Graphs with $\operatorname{odr}(G) \leq 3$

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For each graph G, we have $\operatorname{odr}(G) \leq 3$ if and only if

- *each component is a path, or*
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Proof (Sketch).

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Proof (Sketch).

- Sufficiency: strategies for Builder.
- Necessity: strategies for Painter. Both the greedy $\mathcal{S}_2$-Painter and the greedy $\mathcal{L}$-Painter are used, where $\mathcal{L}$ is the family of linear forests.

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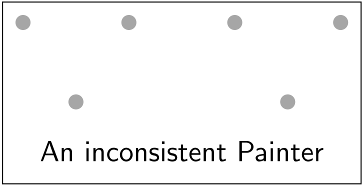

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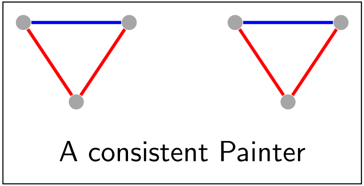

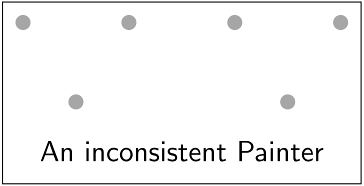

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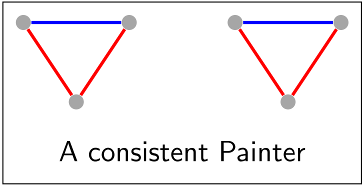

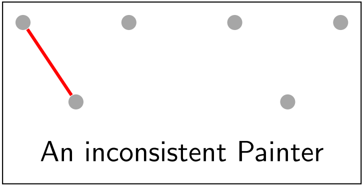

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Theorem (Consistent Painter)

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If $\mathcal{A}$ is a Painter strategy that edge-colors graphs in $\mathcal{H}$, then there exists a consistent Painter strategy $\mathcal{A}'$ that edge-colors graphs in $\mathcal{H}$ such that every edge-colored component produced by $\mathcal{A}'$ is also produced by $\mathcal{A}$.

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- Consistent Painter applies to $\mathcal{H} = \mathcal{S}_k$.

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Theorem (Consistent Painter)

Let $\mathcal{H}_0$ be a family of connected graphs and let $\mathcal{H}$ be the family of graphs that are disjoint unions of members of $\mathcal{H}_0$.
If $\mathcal{A}$ is a Painter strategy that edge-colors graphs in $\mathcal{H}$, then there exists a consistent Painter strategy $\mathcal{A}'$ that edge-colors graphs in $\mathcal{H}$ such that every edge-colored component produced by $\mathcal{A}'$ is also produced by $\mathcal{A}$.

- Consistent Painter applies to $\mathcal{H} = \mathcal{S}_k$.
- When proving upper bounds on $\mathrm{odr}(G)$, it suffices to consider a consistent Painter.

Trees

Theorem

Let T be a tree with a single vertex r of maximum degree. If $d(r) = a$ and $d(u) \leq b$ for each $u \neq r$, then $\mathrm{odr}(T) \leq a + b - 1$.

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Corollary

If T is a tree, then $\mathrm{odr}(T) \leq 2\Delta(T) - 1$.

Trees

Theorem

Let T be a tree with a single vertex r of maximum degree. If $d(r) = 4$ and $d(u) \leq 3$ for each $u \neq r$, then $\operatorname{odr}(T) \leq 4+3-1 = 6$.

Red Tree	Blue Tree

- Build a red tree and a blue tree in parallel.

Trees

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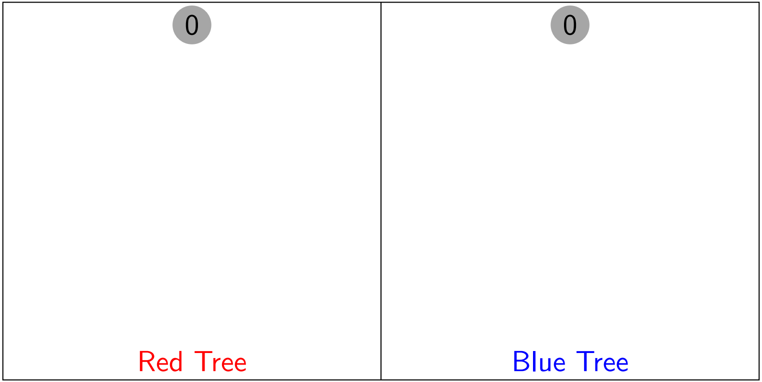
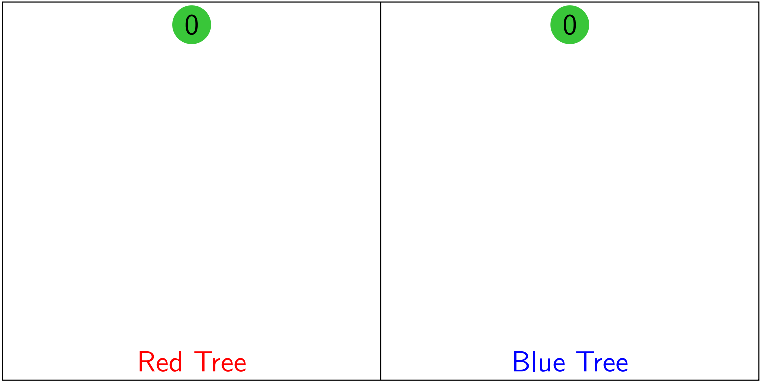


- Build a red tree and a blue tree in parallel.
- Both trees start with fresh vertices to serve as r.

Trees

Theorem

Let T be a tree with a single vertex r of maximum degree. If $d(r) = 4$ and $d(u) \leq 3$ for each $u \neq r$, then $\operatorname{odr}(T) \leq 4 + 3 - 1 = 6$.



- Each tree has an active vertex.
- Builder presents edges between an active vertex and fresh vertices.
- Builder is happy if edges are colored "correctly".

Trees

Theorem

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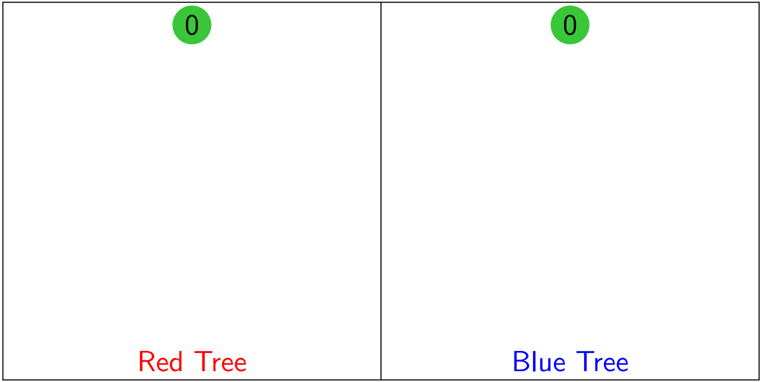

- A tree is *dangerous* when its active vertex has too many (i.e. $b-1 = 2$) children in the "wrong" color.

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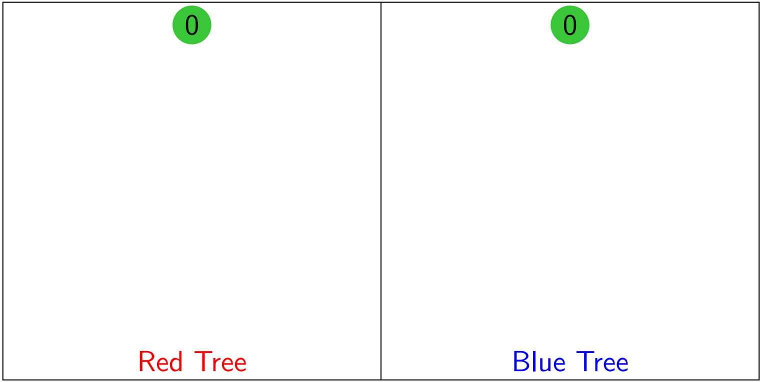

- Present edges until an active vertex is finished or both trees are dangerous.

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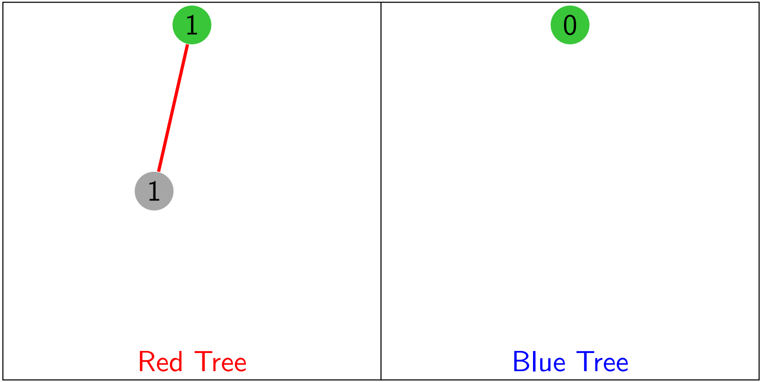

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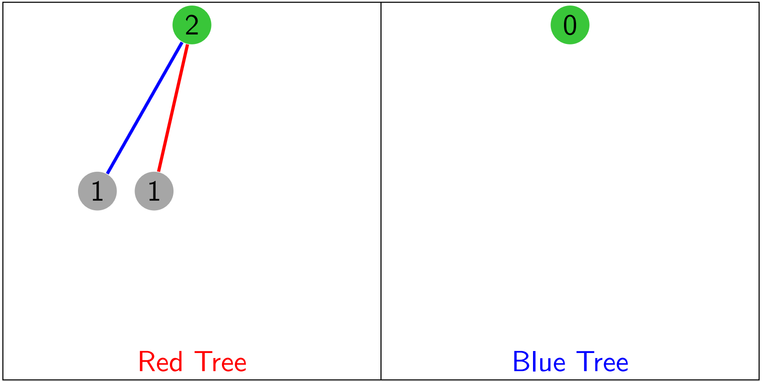

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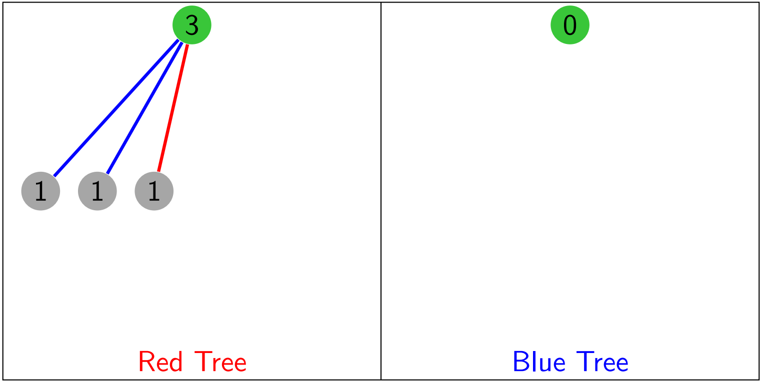

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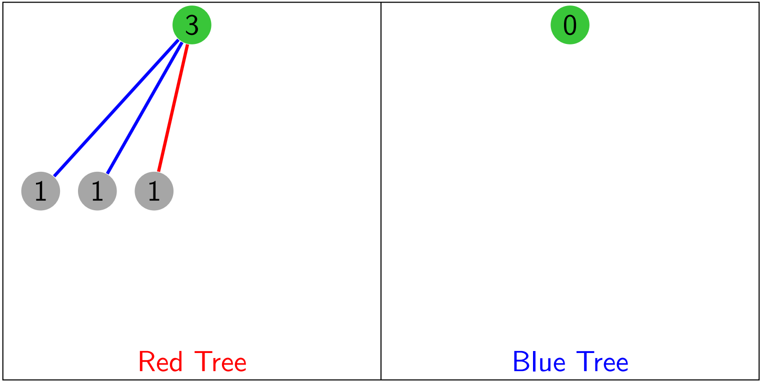

- The red tree is dangerous. Add edges to blue tree.

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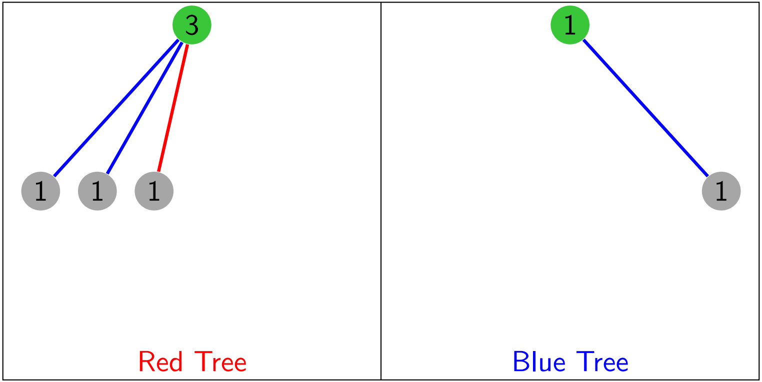

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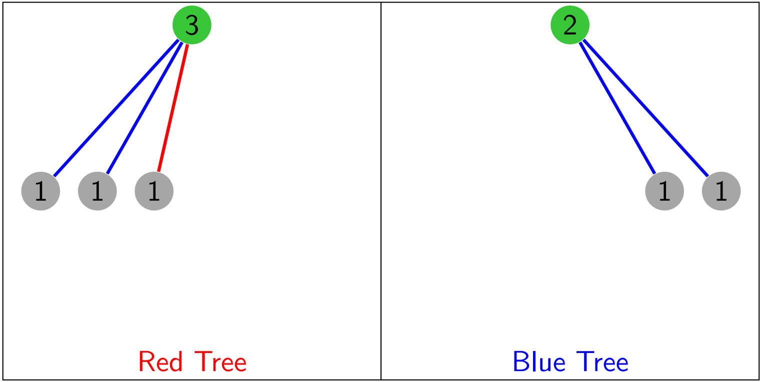

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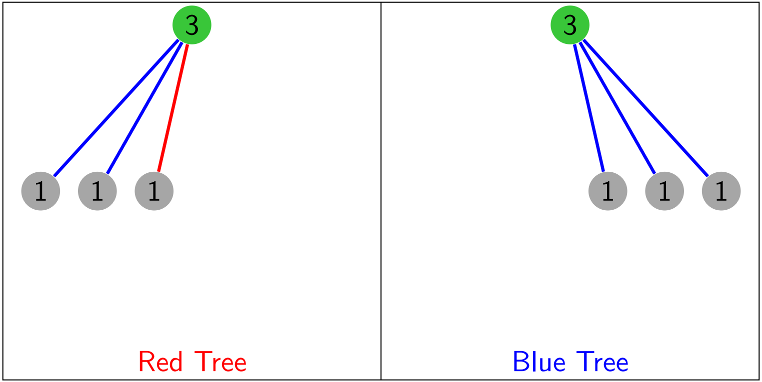

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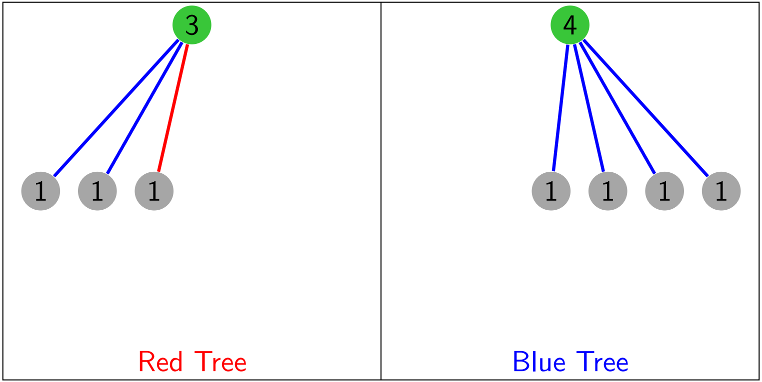

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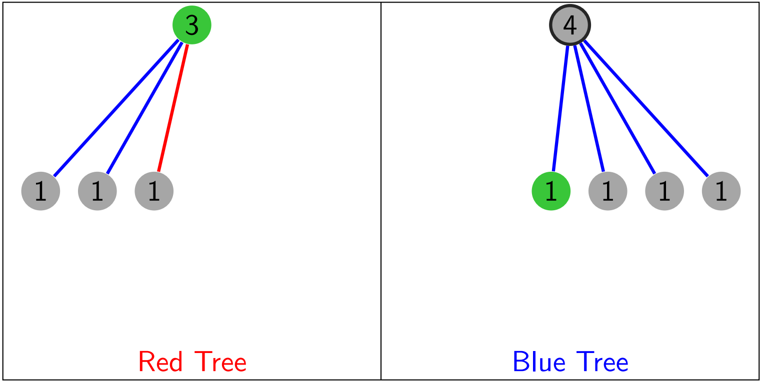

- The red tree is dangerous. Add edges to blue tree.
- The blue active vertex is finished. Move the blue active vertex.

Trees

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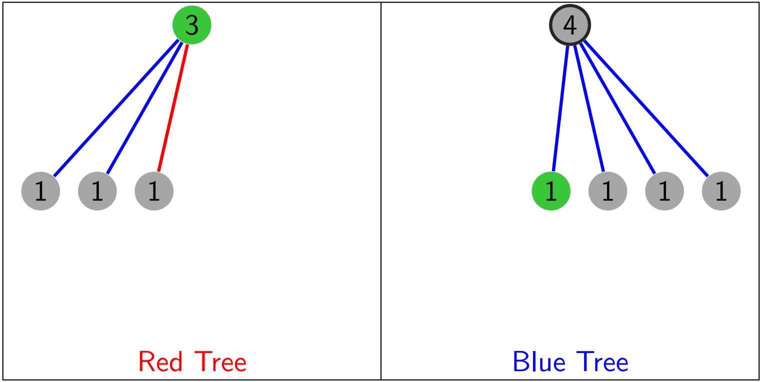

- Status: red tree still dangerous; blue tree not dangerous and blue active vertex not finished.

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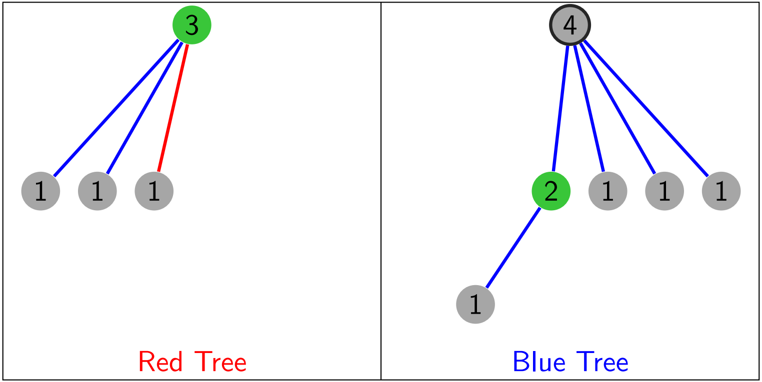

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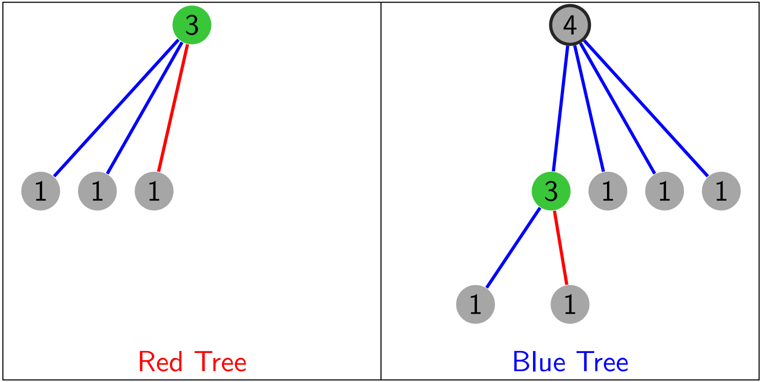

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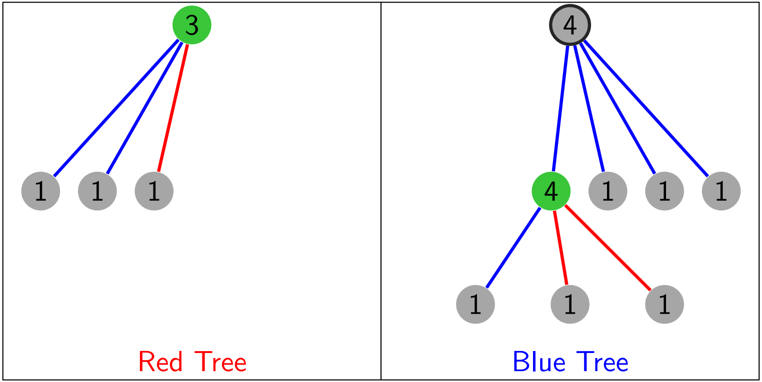

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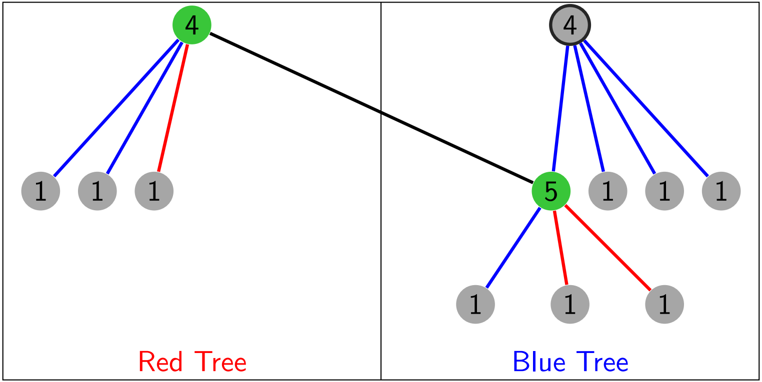

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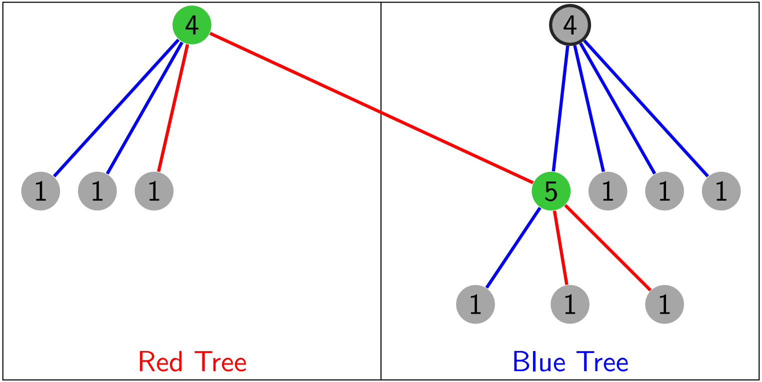

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- Edge colored red, so blue tree's active vertex and red children move to red tree.

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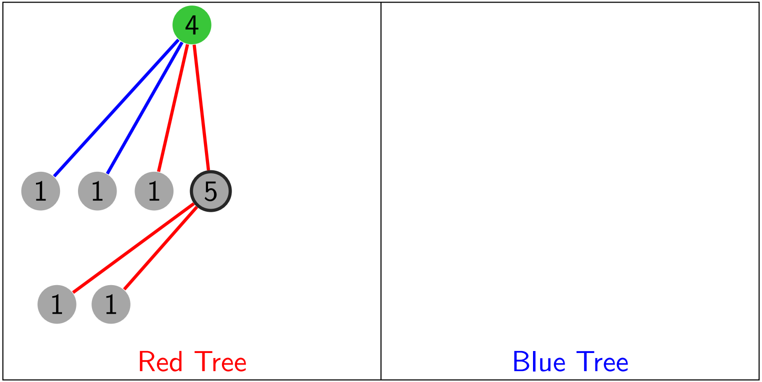

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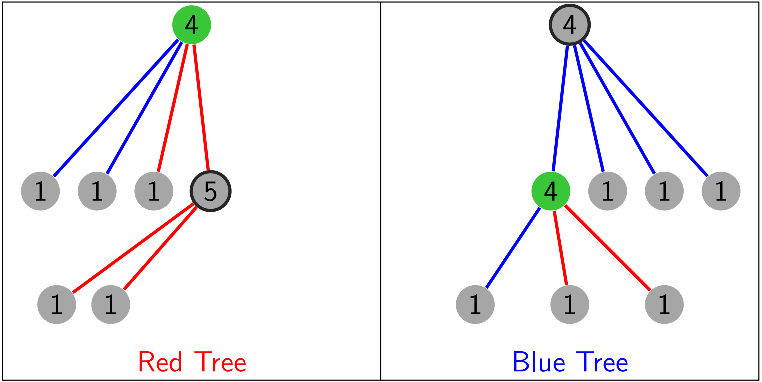

- Use Consistent Painter to regenerate blue tree.

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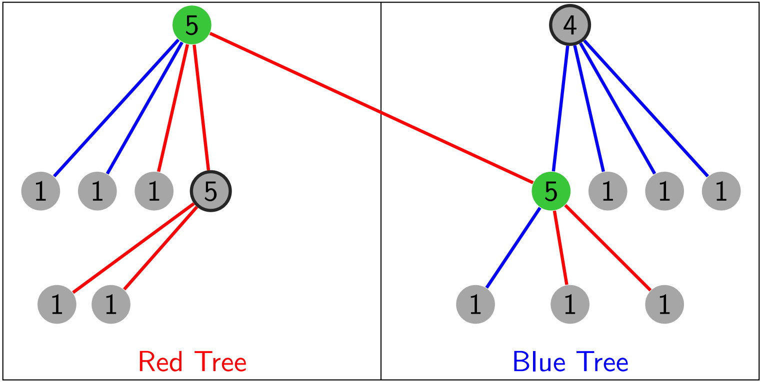

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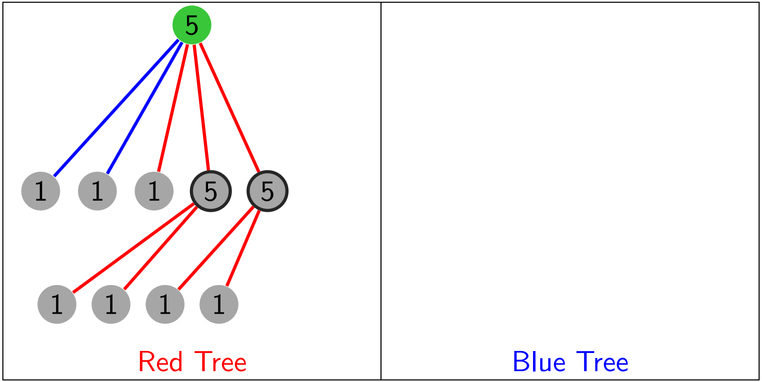

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- Move blue tree's active vertex and red children to red tree.

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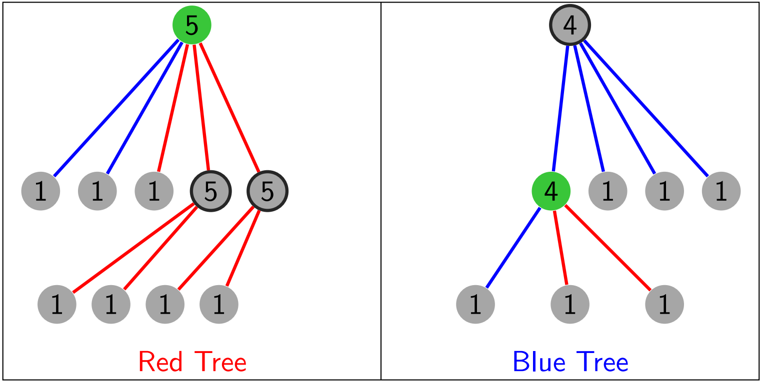

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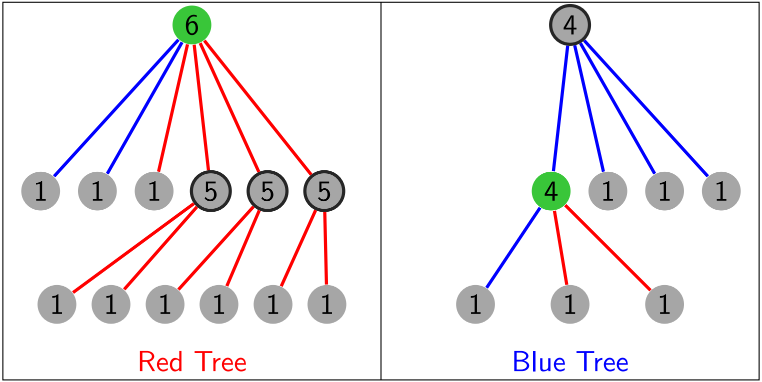

- Use Consistent Painter to regenerate blue tree.
- And once more.

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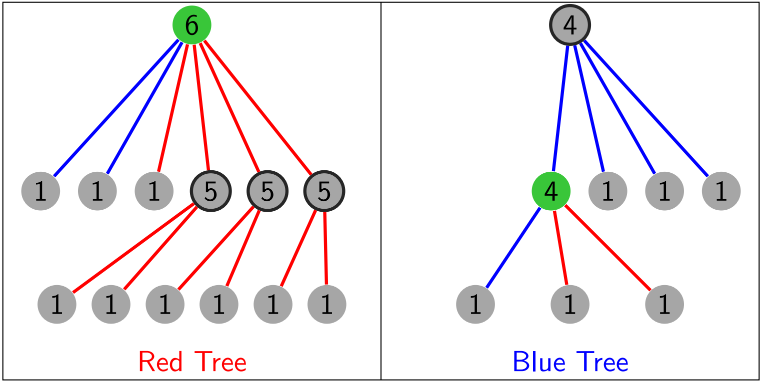

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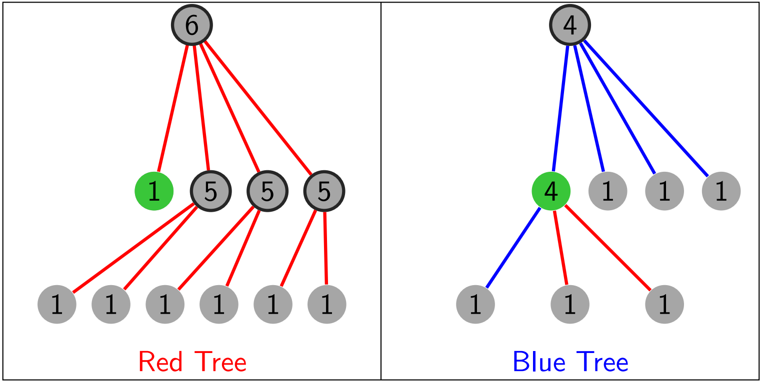

- Edge between active vertices only when both trees are dangerous.
- A tree inherits only finished vertices and leaves from the other.
- Active vertex moves from a finished vertex to a leaf closest to root.

Cycles

Theorem

For each cycle C_n, we have $4 \leq \mathrm{odr}(C_n) \leq 5$.

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Proof (idea).

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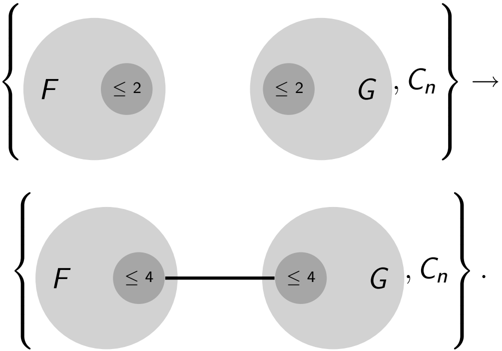
□

Even Cycles

Lemma (Union Lemma)

If n is even, then in $\mathcal{S}_4$, we have



Union Lemma Proof

1. Builder forces $n/2$ copies of $F \cup G$ in one color (say red).

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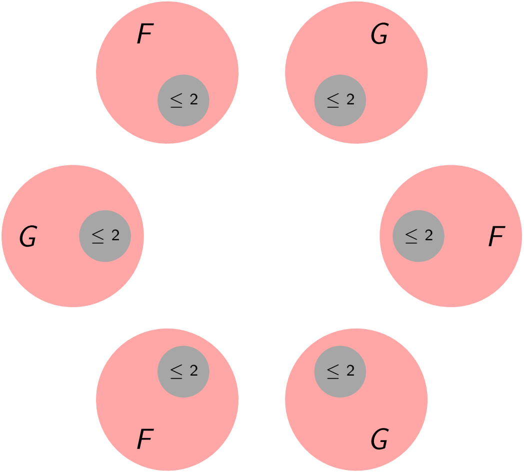

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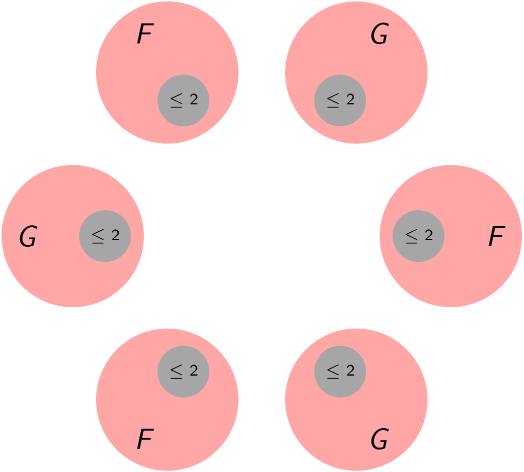

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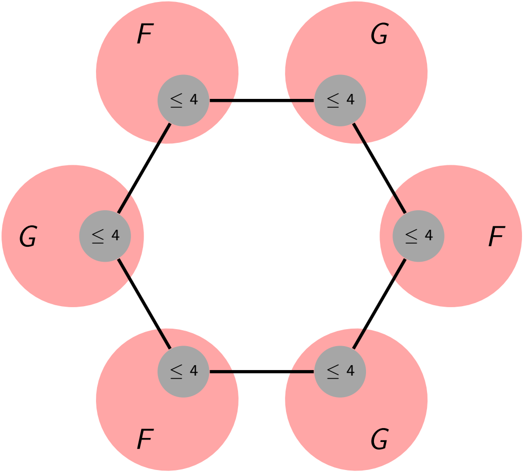

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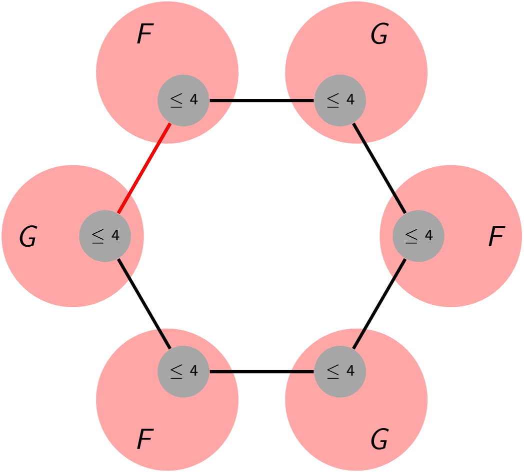

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3. If some edge is red, we have $F \cup G$ in red.

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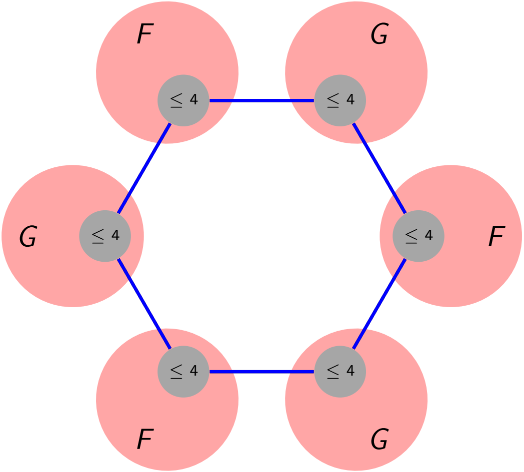

1. Builder forces $n/2$ copies of $F \cup G$ in one color (say red).
2. Builder presents a cycle through specified vertices.
3. If some edge is red, we have $F \cup G$ in red.
4. If all edges are blue, we have C_n in blue.

Odd Cycles

- The Union Lemma does not help when Builder wants to force odd cycles.

Odd Cycles

- The Union Lemma does not help when Builder wants to force odd cycles.
- Nevertheless, weaker variants are possible that help when n is odd.

Theorem

If n is even, $n = 3$, $337 \leq n \leq 514$, or $n \geq 689$, then $\mathrm{odr}(C_n) = 4$.

Open Problems

1. Characterize when $\mathrm{odr}(G) \leq 4$.

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   4.1 Yes for trees: $\mathrm{odr}(T) \leq 2\Delta(T) - 1$.
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5. Develop more strategies for Painter.