Binary Subtrees with Few Path Labels

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University of Illinois at Urbana-Champaign

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Burlington, VT
19 June 2008

History

Rod Downey

Noam Greenberg

Carl Jockusch

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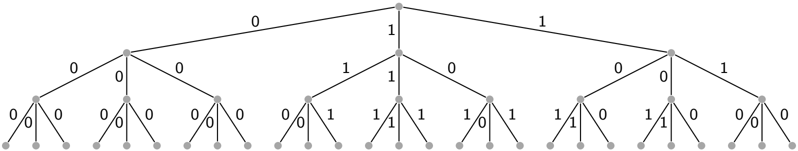
- June 2008: Joint paper in preparation.

The Problem



Ternary tree T with depth $n = 3$.

- Let T be a $\{0, 1\}$-edge-labeled perfect ternary tree of depth n.

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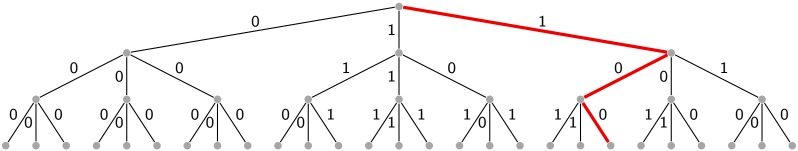

This path has path label 100.

- Let T be a $\{0,1\}$-edge-labeled perfect ternary tree of depth n.
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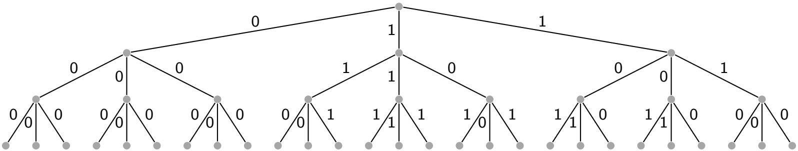

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- Let T be a $\{0,1\}$-edge-labeled perfect ternary tree of depth n.
- Each path from the root to a leaf gives a path label in $\{0,1\}^n$.
- Let $f(T)$ be the min., over all perfect binary subtrees $S \subseteq T$ of depth n, of the number of path labels along paths in S.

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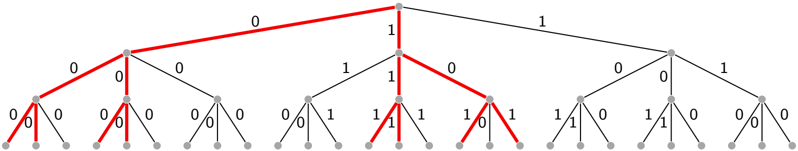

This subtree contains 3 path labels, so $f(T) \leq 3$.

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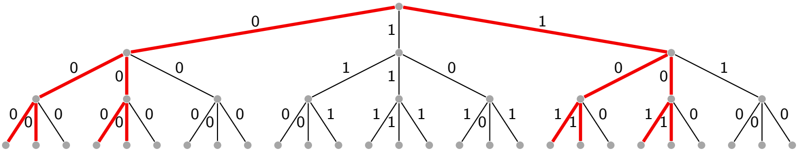

In fact, $f(T) = 2$.

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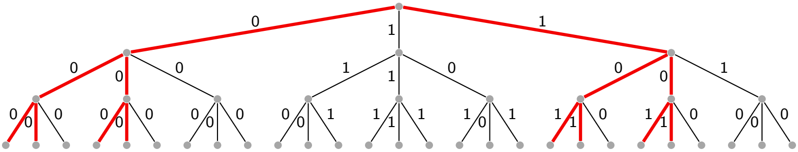

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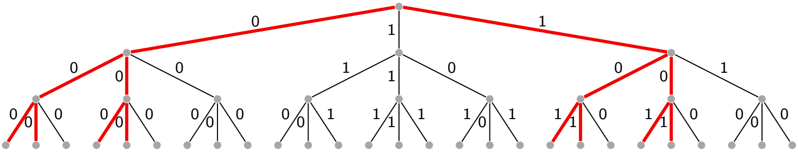

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- Let $f(n)$ be the max., over all $\{0,1\}$-edge-labeled perfect ternary trees T of depth n, of $f(T)$.
- From now on, all trees are perfect and $\{0,1\}$-edge-labeled; all subtrees have full depth.

Main Result

Theorem

There exist positive constants c_1 and c_2 such that

$$2^{\frac{n-3}{\lg 3}} \leq f(n) \leq c_1 2^{n - c_2\sqrt{n}}.$$

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Corollaries

- $\displaystyle\lim_{n\to\infty} \frac{f(n)}{2^n} = 0$
- $1.54856 \approx 2^{\frac{1}{\lg 3}} \leq \displaystyle\lim_{n\to\infty} (f(n))^{1/n} \leq 2$

Preliminaries

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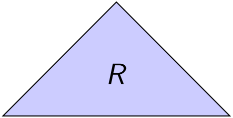

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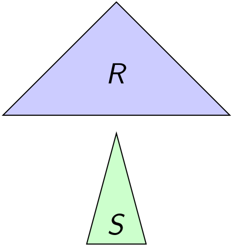

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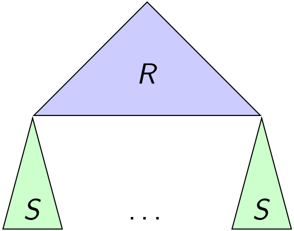

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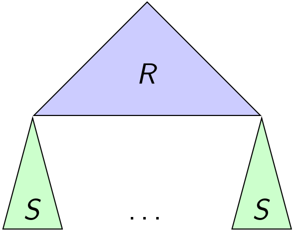

- Let R be a ternary tree of depth r which maximizes $f(R)$.
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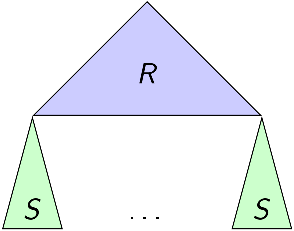

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Corollary

$$\lim_{n\to\infty}(f(n))^{1/n} = \sup\left\{(f(n))^{1/n} \mid n \geq 1\right\}$$

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- The construction uses two different kinds of trees.

Lower Bound: Construction of R_n

Proposition

Let $a_0 = 1$ and $a_n = \lceil 3a_{n-1}/2 \rceil$ for $n \geq 0$. If $n \geq 0$, then there exists a ternary tree R_n of depth n in which each path label occurs at most a_n times.

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- Of the $3|L_x|$ edges in R_n below vertices in L_x, arbitrarily choose $\lceil 3|L_x|/2 \rceil$ to have label 0; the remaining $\lfloor 3|L_x|/2 \rfloor$ edges are labeled 1.

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In R_n, a path label $x \in \{0,1\}^n$ occurs at most $\lceil 3a_{n-1}/2 \rceil$ times.

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Corollary

If $n \geq 0$, then there exists a ternary tree R_n of depth n in which each path label occurs at most $2\left(\frac{3}{2}\right)^n$ times.

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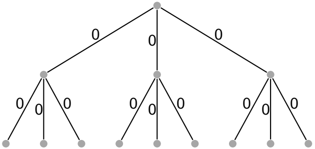

Q_{00}

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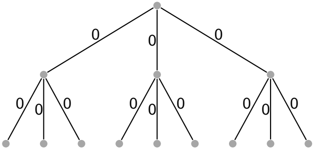

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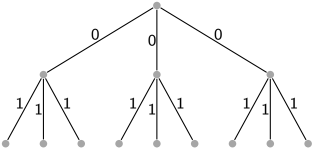

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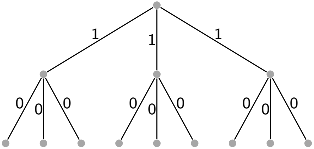

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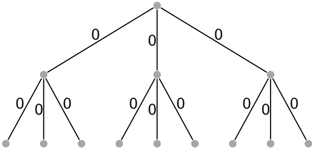

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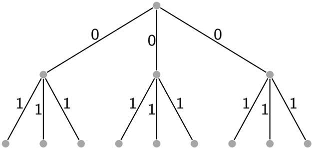

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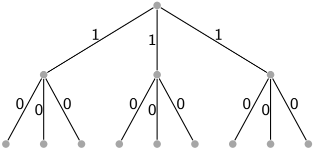

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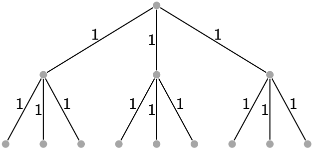

Q_{11}

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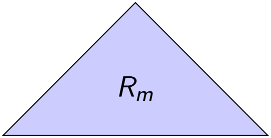

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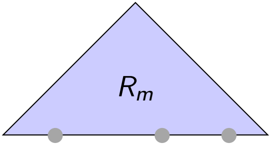

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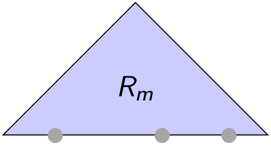

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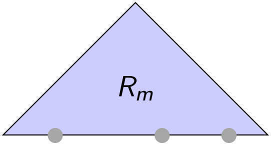

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- For each $u \in L_x$, arbitrarily choose a distinct bitstring $y(u) \in \{0,1\}^s$.
- Because $|L_x| \leq 2\left(\frac{3}{2}\right)^m \leq 2^s$, enough bitstrings are available.

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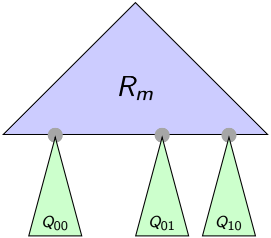

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- At each $u \in L_x$, attach a copy of $Q_{y(u)}$.

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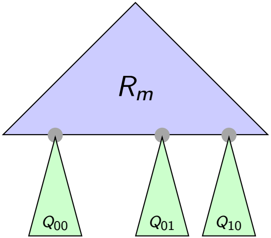

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- Because $|L_x| \leq 2\left(\frac{3}{2}\right)^m \leq 2^s$, enough bitstrings are available.
- At each $u \in L_x$, attach a copy of $Q_{y(u)}$.
- Repeat for each $x \in \{0,1\}^m$.

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Proof (sketch).

Either n or $n-1$ is of the form $m + \left\lceil \lg 2 \left(\frac{3}{2}\right)^m \right\rceil$ for some integer m, in which case the Lemma applies. □

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Corollary

$$\lim_{n \to \infty} (f(n))^{1/n} \geq 2^{\frac{1}{\lg 3}} \approx 1.54856$$

Upper Bound: Overview

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- Upper bound uses several lemmas.

Upper Bound: Monochromatic Subtree Lemma

Lemma (Monochromatic Subtree Lemma; folklore?)

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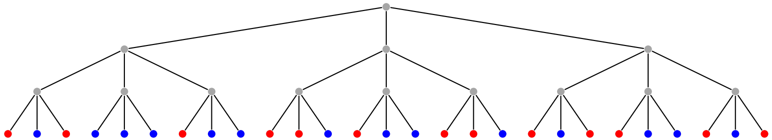

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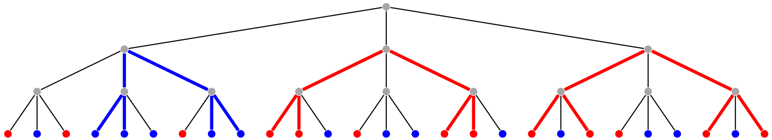

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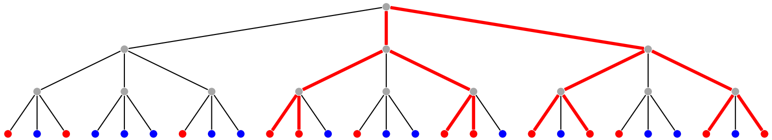

□

Upper Bound: Orthogonal Partitions

- Let Υ be a finite ground set.

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- A pair of partitions $\{X, \overline{X}\}$ and $\{Y, \overline{Y}\}$ of Υ is α-orthogonal if all four of the cross intersections $X \cap Y$, $X \cap \overline{Y}$, $\overline{X} \cap Y$, and $\overline{X} \cap \overline{Y}$ have size at least $\alpha \frac{|\Upsilon|}{4}$.

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- A family of partitions $\mathcal{F}$ of Υ is α-orthogonal if each pair of (distinct) partitions in Υ is α-orthogonal.

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Lemma (Orthogonal Family Lemma)

If $|\Upsilon| = t$ *and* $0 \leq \alpha \leq 1$, *then there exists an* α*-orthogonal family of partitions* $\mathcal{F}$ *of* Υ *with*

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- Chernoff bound: $\mathcal{F}$ is α-orthogonal with positive probability.

□

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Let $T_1, T_2, \ldots, T_k$ *be ternary trees of depth* n *and let* $\Upsilon = \{0, 1\}^n$*.*

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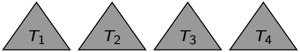

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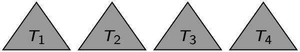

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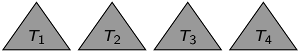

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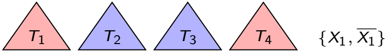

- Consider a ptn. $\{X_1, \overline{X_1}\} \in \mathcal{F}$.
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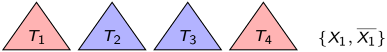

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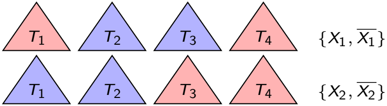

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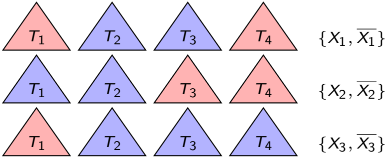

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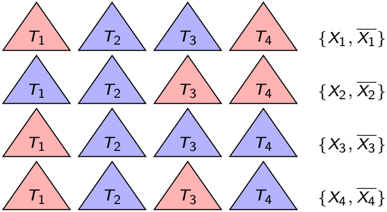

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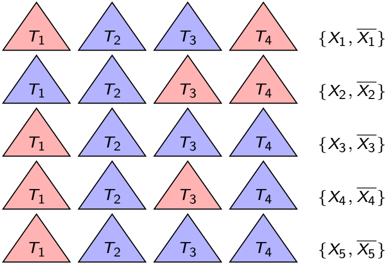

- Repeat for each ptn. in $\mathcal{F}$.
- $\mathcal{F}$ is large, so some pair $\{X, \overline{X}\}$ and $\{Y, \overline{Y}\}$ give the same red/blue ptn. of $\{T_1, \ldots, T_k\}$.

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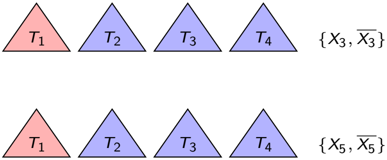

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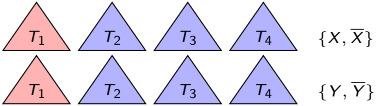

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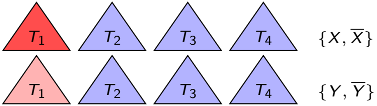

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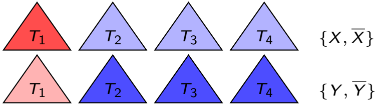

- If T_j is red under $\{X, \overline{X}\}$, then T_j has a binary subtree S_j in which every path label is in X.
- If T_j is blue under $\{Y, \overline{Y}\}$, then T_j has a binary subtree S_j in which every path label is in $\overline{Y}$.

□

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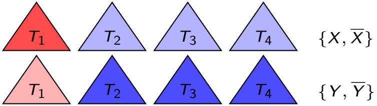

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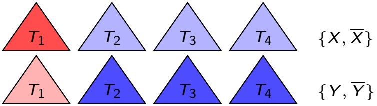

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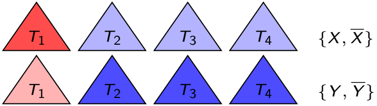

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- $\mathcal{F}$ is α-orthogonal: $|\overline{X} \cap Y| \geq \frac{\alpha}{4} 2^n$.

□

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Let $T_1, \ldots, T_k$ be ternary trees of depth $n \geq 6 + \lg k$, and let $\Upsilon = \{0, 1\}^n$. There exist binary subtrees $S_1, \ldots, S_k$ with $S_j \subseteq T_j$ such that

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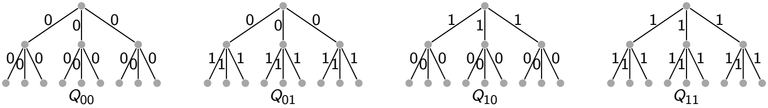
The assumption $n \geq 6 + \lg k$ is tight up to an additive constant. Indeed, if $k = 2^n$:



Upper Bound

Theorem

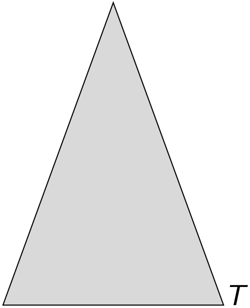
Let $c_1 = \sqrt{\lg(16/15)} \approx 0.3051$ *and* $c_2 = 2^{c_1\sqrt{540}-1} \approx 68.156$. *If* $n \geq 0$, *then* $f(n) \leq c_2 2^{n-c_1\sqrt{n}}$.

Upper Bound

Theorem

Let $c_1 = \sqrt{\lg(16/15)} \approx 0.3051$ *and* $c_2 = 2^{c_1\sqrt{540}-1} \approx 68.156$*. If* $n \geq 0$*, then* $f(n) \leq c_2 2^{n - c_1\sqrt{n}}$*.*

Proof (sketch).



- Let T be a ternary tree with depth n.

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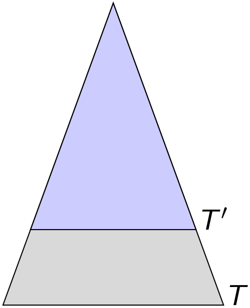

- Let T be a ternary tree with depth n.
- Let T' be the ternary subtree of T up to depth $m \approx n - c_1\sqrt{n}$.

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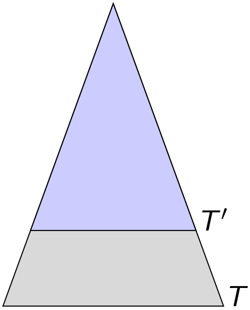

- Let T be a ternary tree with depth n.
- Let T' be the ternary subtree of T up to depth $m \approx n - c_1\sqrt{n}$.
- Obtain a binary subtree $S' \subseteq T'$ that uses few path labels.

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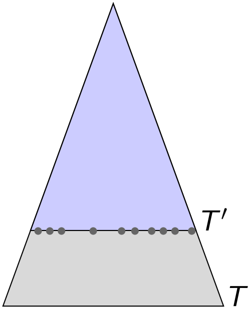

- Fix $x \in \{0,1\}^m$ and let L_x be the set of leaves in S' that are endpoints of a path with path label x.

□

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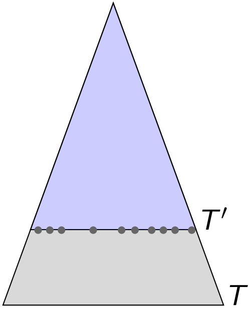

- Fix $x \in \{0,1\}^m$ and let L_x be the set of leaves in S' that are endpoints of a path with path label x.
- Two cases: if L_x is large, then extend S' at vertices in L_x arbitrarily.

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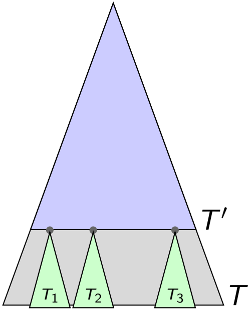

- Fix $x \in \{0,1\}^m$ and let L_x be the set of leaves in S' that are endpoints of a path with path label x.
- Two cases: if L_x is large, then extend S' at vertices in L_x arbitrarily.
- If L_x is small, apply Binary Subtrees Lemma (2) to extend S' at vertices in L_x. □

Summary & Open Problems

Theorem

There exist positive constants c_1 and c_2 such that

$$2^{\frac{n-3}{\lg 3}} \leq f(n) \leq c_1 2^{n - c_2\sqrt{n}}.$$

Corollary

$$1.54856 \approx 2^{\frac{1}{\lg 3}} \leq \lim_{n\to\infty} (f(n))^{1/n} \leq 2.$$

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- Is it true that $\lim_{n\to\infty}(f(n))^{1/n} < 2$?
- For each $p < q$, consider the analogous problem on $\{0, 1, \ldots, p-1\}$-edge-labeled perfect q-ary trees. Nothing is known except our results for $(p, q) = (2, 3)$.