FIRST-FIT IS LINEAR ON POSETS EXCLUDING TWO LONG INCOMPARABLE CHAINS

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Abstract. A poset is ($\underline{r}+\underline{s}$)-free if it does not contain two incomparable chains of size r and s, respectively. We prove that when r and s are at least 2, the First-Fit algorithm partitions every ($\underline{r}+\underline{s}$)-free poset P into at most $8(r-1)(s-1)w$ chains, where w is the width of P. This solves an open problem of Bosek, Krawczyk, and Szczypka (*SIAM J. Discrete Math., 23(4):1992–1999, 2010*).

1. Introduction

A *chain* in a poset is a set of elements that are pairwise comparable, and an *antichain* is a set of elements that are pairwise incomparable. The *height* of a poset is the size of a largest chain, and the *width* is the size of a largest antichain. In the *on-line chain partitioning problem*, the elements of an unknown poset P are revealed one by one in some order. Each time a new element x is presented, one has to assign a color to x, maintaining the property that each color class is a chain. The goal is to minimize the number of chains in the resulting chain partition of P.

This classical problem has received increased attention in the recent years; see, for example, the survey by Bosek, Felsner, Kloch, Krawczyk, Matecki, and Micek [1]. In this context, the quality of a solution is typically compared against the width w of P. Since elements of an antichain must receive distinct colors, at least w colors are needed. By Dilworth's theorem, if all elements of P are presented before any are colored, then w colors suffice. In the on-line setting, more colors are needed.

Let $\mathrm{val}(w)$ be the least k such that there is an on-line algorithm that partitions posets of width w into at most k chains. Establishing that $\mathrm{val}(w)$ is finite when $w \geq 2$ is challenging. In 1981, Kierstead [9] proved that $\mathrm{val}(w) \leq (5^w - 1)/4$. For nearly three decades, Kierstead's result was the

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best known upper bound on $\mathrm{val}(w)$. Recently, Bosek and Krawczyk [2] showed that $\mathrm{val}(w) \leq w^{16 \lg w}$ (see [1] for a proof sketch). From below, Szemerédi proved that $\mathrm{val}(w) \geq \binom{w+1}{2}$ (see [1, 9]), and Bosek *et al.* [1] showed that $\mathrm{val}(w) \geq (2 - o(1))\binom{w+1}{2}$. One of the central questions in the theory of on-line problems on partial orders is whether $\mathrm{val}(w)$ is bounded above by a polynomial in w.

In this paper, we are interested in the performance of an on-line chain partitioning algorithm called First-Fit. Using the positive integers for colors, First-Fit colors x with the least j such that x and all elements previously assigned color j form a chain. It is known that, for general posets, the number of chains used by First-Fit is not bounded by a function of w. In fact, Kierstead [9] showed that First-Fit uses arbitrarily many chains on posets of width 2 (see also [4]).

Nevertheless, First-Fit performs well on certain classes of posets, such as interval orders. An *interval order* is a poset whose elements are closed intervals on the real line, with $[a, b] < [c, d]$ if and only if $b < c$. Let $\mathrm{FF}(w)$ be the maximum number of chains that First-Fit uses on interval orders of width w. Kierstead [10] proved that $\mathrm{FF}(w) \leq 40w$. Kierstead and Qin [11] subsequently improved the bound, showing that $\mathrm{FF}(w) \leq 25.8w$. Later, Pemmaraju, Raman, and Varadarajan [16] (see also [17]) proved that $\mathrm{FF}(w) \leq 10w$ with an elegant argument known as the Column Construction Method. Their proof was later refined by Brightwell, Kierstead, and Trotter [5] and independently by Narayanaswamy and Babu [15] to show that $\mathrm{FF}(w) \leq 8w$.

From early results of Kierstead and Trotter [14], it follows that $\mathrm{FF}(w) \geq (3+\varepsilon)w$ for some positive ε. Chrobak and Ślusarek [6] showed that $\mathrm{FF}(w) \geq 4w - 9$ when $w \geq 4$ and subsequently improved the multiplicative constant to 4.45 at the expense of a weaker additive constant. In 2004, Kierstead and Trotter [13] proved that $\mathrm{FF}(w) \geq 4.99w - c$ for some constant c with the aid of a computer. Recently, Kierstead, Smith, and Trotter [12] proved that for each positive ε, there is a constant c such that $\mathrm{FF}(w) \geq (5 - \varepsilon)w - c$.

If P and Q are posets, then $P+Q$ denotes the poset obtained from disjoint copies of P and Q where elements in the copy of P are incomparable to elements in the copy of Q. A poset P is *Q-free* if no induced subposet of P is isomorphic to Q. We denote by $\underline{r}$ the poset consisting of a chain of size r. Fishburn [8] characterized the interval orders as the posets that are $(\underline{2} + \underline{2})$-free. When r and s are at least two, the family of $(\underline{r} + \underline{s})$-free posets contains the family of interval orders. Bosek, Krawczyk, and Szczypka [4] showed that when $r \geq s$, First-Fit partitions every $(\underline{r} + \underline{s})$-free poset into at most $(3r - 2)(w - 1)w + w$ chains. They asked whether First-Fit uses only a linear number of chains, in terms of w, on $(\underline{r}+\underline{s})$-free posets, as it does on interval orders. This question also appears in the survey of Bosek *et al.* [1] and in a recent paper of Felsner, Krawczyk, and Trotter [7].

We give a positive answer to this question by showing that First-Fit partitions every $(\underline{r}+\underline{s})$-free poset into at most $8(r-1)(s-1)w$ chains. As far as we know, this also provides the first proof that some on-line algorithm uses $o(w^2)$ chains on $(\underline{r}+\underline{s})$-free posets. Our proof is strongly influenced by the Column Construction Method of Pemmaraju *et al.* [17] and can be viewed as a generalization of that technique from interval orders to $(\underline{r}+\underline{s})$-free posets.

In Section 2, we present our generalization of the Column Construction Method and establish several of its properties. In Section 3, we combine these results with a structural lemma about $(\underline{r}+\underline{s})$-free posets to obtain our main result.

2. Evolution of Societies

Let P be a poset. A *First-Fit chain partition* is an ordered partition $C_1, \ldots, C_m$ of P into non-empty chains such that if $i < j$ and $x \in C_j$, then some element in C_i is incomparable to x. Note that if $C_1, \ldots, C_m$ is a First-Fit chain partition, then First-Fit produces this partition when elements in C_1 are presented first, followed by elements in C_2, and continuing through elements in C_m. Conversely, every ordered partition produced by First-Fit is a First-Fit chain partition.

A *group* is a set of elements in P. A *t-society* is a pair (S, F) where S is a set of groups and F is a *friendship function* from $S \times [t]$ to $S \cup \{\star\}$, where $[t]$ denotes the set $\{1, \ldots, t\}$. Each group $X \in S$ has slots for up to t friends. We say that *X lists Y as a friend in slot k* if $F(X, k) = Y$. It is possible that X does not list any friend in slot k, in which case $F(X, k) = \star$.

The overview of our proof is as follows. Given an $(\underline{r}+\underline{s})$-free poset P, we first exploit the structure of P to define an initial t-society (S_0, F_0) for some t depending on s. Next, we fix a First-Fit chain partition $C_1, \ldots, C_m$, which we extend to an infinite sequence of chains by defining $C_j = \varnothing$ for $j > m$. We allow the initial t-society to evolve, generating a sequence of t-societies $(S_0, F_0), \ldots, (S_n, F_n)$. For $j \geq 1$, the t-society (S_j, F_j) is obtained from (S_{j-1}, F_{j-1}) by following certain rules that depend on C_j and the previous transitions. It is helpful to view the t-societies as vertices of a path and to associate the edge joining (S_{j-1}, F_{j-1}) and (S_j, F_j) with the chain C_j.

During the evolution, we maintain that $S_0 \supseteq S_1 \supseteq \cdots \supseteq S_n$. The evolution ends when a t-society (S_n, F_n) is generated where $S_n = \varnothing$. The proof proceeds in two parts. First, we show that a long evolution implies that some group in the initial t-society is large. Second, given an $(\underline{r}+\underline{s})$-free poset P, we show how to construct an initial t-society of groups inducing subposets of height at most $r-1$ that leads to a long evolution. Because large posets of bounded height contain large antichains, we obtain a lower bound on the width of P.

In our societies, friendship is a lifetime commitment: if $F_{j-1}(X,k)=Y$ and $\{X,Y\}\subseteq S_j$, then $F_j(X,k)=Y$. If X survives the transition from S_{j-1} to S_j but Y does not, then X either chooses a new friend for its kth slot or leaves its kth slot empty according to the rules of a *replacement scheme*. We postpone the presentation of the details of our replacement scheme and the construction of the initial t-society.

A group X may survive the transition from S_{j-1} to S_j in three ways, each of which defines a transition type. We use the first three Greek letters α, β, and γ to name the transition types. When $a\in\{\alpha,\beta,\gamma\}$ and $i\le j$, we define $N^a_{i,j}(X)$ to be the number of transitions of type a that X makes in the evolution from (S_i,F_i) to (S_j,F_j).

Let $\varepsilon=1/2t$; in Lemma 2.4, we will find that this choice of ε is optimal. We now describe the rules that govern which groups survive the jth transition from S_{j-1} to S_j. Let X be a group in S_{j-1}.

(1) If X has non-empty intersection with C_j, then X makes an α-transition from S_{j-1} to S_j.
(2) Otherwise, if some friend of X in the t-society (S_{j-1},F_{j-1}) has non-empty intersection with C_j, then X makes a β-transition from S_{j-1} to S_j.
(3) Otherwise, if there is an i such that $N^\alpha_{i,j-1}(X)>\varepsilon(j-i)$, then X makes a γ-transition from S_{j-1} to S_j.

If none of the three rules apply, then $X\notin S_j$, and other groups that list X as a friend and survive to S_j update their list of friends according to the replacement scheme.

First, we show that a long evolution implies that some group is large. We need several lemmas.

Lemma 2.1. *Fix an evolution $(S_0,F_0),\dots,(S_n,F_n)$ of t-societies. Let $Y_1,Y_2,\dots,Y_q$ be a list of groups and let $[a_1,b_1],\dots,[a_q,b_q]$ be a sequence of disjoint intervals with integral endpoints in $[0,n]$ such that b_j is the largest integer such that $Y_j\in S_{b_j}$. The sum $\sum_{j=1}^q N^\alpha_{a_j,b_j}(Y_j)$ is at most εn.*

Proof. If $a_j=b_j$, then clearly $N^\alpha_{a_j,b_j}(Y_j)=0$. Hence, we may assume that $0\le a_1<b_1<\cdots<a_q<b_q$. Also, $b_q<n$ because $S_n=\varnothing$. Note that $Y_j\notin S_{b_j+1}$. It follows that Y_j did not satisfy the third condition in the transition from S_{b_j} to S_{b_j+1} and therefore $N^\alpha_{a_j,b_j}(Y_j)\le\varepsilon(b_j+1-a_j)$. Also,

$\sum_{j=1}^{q}(b_j + 1 - a_j) \leq n$ because the intervals $[a_j, b_j]$ are disjoint subsets of $[0, n-1]$ with integral endpoints. $\square$

Our next lemma provides a bound on the number of β-transitions that a group can make if it survives to the last non-empty t-society.

Lemma 2.2. *Fix an evolution $(S_0, F_0), \ldots, (S_n, F_n)$ of t-societies. If $X \in S_{n-1}$, then $N^{\beta}_{0,n-1}(X) \leq t\varepsilon n$.*

Proof. Let $X \in S_{n-1}$, and for each $k \in [t]$, let $\mathcal{Y}_k$ be the set of groups that X lists as a friend in slot k at some point in the evolution. If X makes a β-transition from S_{j-1} to S_j, then there is a slot k and group $Y \in \mathcal{Y}_k$ such that $F_{j-1}(X, k) = Y$ and Y has non-empty intersection with C_j. Because $Y \in S_{j-1}$ and Y has non-empty intersection with C_j, we have that Y makes an α-transition from S_{j-1} to S_j. It follows that

$$N^{\beta}_{0,n-1}(X) \leq \sum_{k=1}^{t} \sum_{Y \in \mathcal{Y}_k} N^{\alpha}_{I(Y)}(Y)$$

where $I(Y)$ is denotes the interval during which X lists Y as a friend. (Formally, $j \in I(Y)$ if and only if $F_j(X, k) = Y$ for some $k \in [t]$.) It suffices to show that $\sum_{Y \in \mathcal{Y}_k} N^{\alpha}_{I(Y)}(Y) \leq \varepsilon n$ for each $k \in [t]$. Because $\{I(Y) \colon Y \in \mathcal{Y}_k\}$ are disjoint intervals, the bound follows from Lemma 2.1. $\square$

Next, we show that for each group X, the α-transitions that X makes constitute a large fraction of the total number of X's transitions not of type β.

Lemma 2.3. *Fix an evolution $(S_0, F_0), \ldots, (S_n, F_n)$ of t-societies. If X is a group, then $N^{\alpha}_{0,j}(X) \geq \varepsilon(N^{\alpha}_{0,j}(X) + N^{\gamma}_{0,j}(X))$ for each j with $X \in S_j$.*

Proof. If $j = 0$, then the inequality holds. For $j \geq 1$, the inequality holds immediately by induction unless X makes a γ-transition from S_{j-1} to S_j. In this case, there is some i such that $N^{\alpha}_{i,j-1}(X) > \varepsilon(j - i)$. Applying the inductive hypothesis to obtain a lower bound on $N^{\alpha}_{0,i}(X)$, it follows that

$$\begin{aligned}
N^{\alpha}_{0,j}(X) &= N^{\alpha}_{0,i}(X) + N^{\alpha}_{i,j-1}(X) \\
&\geq \varepsilon(N^{\alpha}_{0,i}(X) + N^{\gamma}_{0,i}(X)) + \varepsilon(j - i) \\
&\geq \varepsilon(N^{\alpha}_{0,i}(X) + N^{\gamma}_{0,i}(X)) + \varepsilon(N^{\alpha}_{i,j}(X) + N^{\gamma}_{i,j}(X)) \\
&= \varepsilon(N^{\alpha}_{0,j}(X) + N^{\gamma}_{0,j}(X))
\end{aligned}$$

as required. $\square$

We are now able to show that a long evolution implies that some group is large.

Lemma 2.4. *Fix an evolution $(S_0, F_0), \dots, (S_n, F_n)$ of t-societies. If $X \in S_{n-1}$, then $|X| \geq (n-2)/4t$.*

Proof. Whenever X makes an α-transition from S_{j-1} to S_j, it has non-empty intersection with chain C_j. Because the chains are disjoint, it follows that $|X| \geq N^\alpha_{0,n-1}(X)$. By Lemma 2.3, we have that $N^\alpha_{0,n-1}(X) \geq \varepsilon(N^\alpha_{0,n-1}(X) + N^\gamma_{0,n-1}(X))$. Note that X makes $n-1$ transitions in total, because $X \in S_{n-1}$. Hence $N^\alpha_{0,n-1}(X) + N^\beta_{0,n-1}(X) + N^\gamma_{0,n-1}(X) = n-1$. By Lemma 2.2, we have that $N^\alpha_{0,n-1}(X) + t\varepsilon n + N^\gamma_{0,n-1}(X) \geq n-1$. Consequently, $N^\alpha_{0,n-1}(X) \geq \varepsilon(n-1-t\varepsilon n)$. With $\varepsilon = 1/2t$, we obtain $N^\alpha_{0,n-1}(X) \geq (n-2)/4t$ as required. ☐

3. The Initial Society and Replacement Scheme

It remains to describe the initial t-society and our replacement scheme. Both depend on the following structural lemma about $(\underline{r} + \underline{s})$-free posets. The *height* of an element x, denoted $h(x)$, is the size of a largest chain with maximum element x.

Lemma 3.1. *Let r and s be integers with $r \geq 2$ and $s \geq 2$, and let P be an $(\underline{r}+\underline{s})$-free poset. There is a function I which assigns to each element $x \in P$ a non-empty set of consecutive integers $I(x)$ with the following properties.*

1. *For each integer k, the set $\{x \in P\colon\ k \in I(x)\}$ induces a subposet of height at most $r-1$.*
2. *If x and y are incomparable in P, then either $I(x)$ and $I(y)$ have non-empty intersection, or at most $s-2$ integers are strictly between $I(x)$ and $I(y)$.*

Proof. Let q be the height of P. For each $x \in P$, let $Z(x)$ be the set of all elements z such that P contains a chain of size r with minimum element x and maximum element z. When $Z(x)$ is non-empty, define $b(x)$ to be the minimum height of an element in $Z(x)$; we set $b(x) = q+1$ when $Z(x) = \varnothing$. Let $I(x) = \{h(x), \dots, b(x)-1\}$.

Fix an integer k and let $X = \{x \in P\colon\ k \in I(x)\}$. Suppose for a contradiction that X contains a chain $x_1 < \cdots < x_r$. Since $x_r \in X$, we have that $k \in I(x_r)$, which implies that $h(x_r) \leq k$. Similarly, $k \in I(x_1)$ and therefore $k \leq b(x_1) - 1$. Since $x_r \in Z(x_1)$, it follows that $b(x_1) \leq h(x_r)$. Hence $h(x_r) \leq k \leq h(x_r) - 1$, a contradiction. It follows that (1) holds.

It remains to check (2). Suppose that x and y are incomparable. If $I(x)$ and $I(y)$ have non-empty intersection, then (2) holds. Hence, we may assume that every integer in $I(x)$ is less than every integer in $I(y)$. Let i be the greatest integer in $I(x)$ and let j be the least integer in $I(y)$, and note that $i < j \leq q$. Since $i \in I(x)$ but $i+1 \notin I(x)$, it follows that $b(x) - 1 = i$.

Because $i < q$, it follows that $b(x) = i+1 \le q$ and therefore $Z(x) \neq \varnothing$. Hence, there is a chain $x = x_1 < \cdots < x_r$ in P with $h(x_r) = i+1$. Similarly, $h(y) = j$ and there is a chain $y = y_j > \cdots > y_1$ in P with $h(y_k) = k$ for each $k \in [j]$. Let $X = \{x_1, \ldots, x_r\}$ and let $Y = \{y_{i+1}, \ldots, y_j\}$. We claim that every element in X is incomparable to every element in Y. If $x_a \le y_b$, then transitivity implies that $x = x_1 \le y_j = y$, contrary to the assumption that x and y are incomparable. Conversely, if $y_a \le x_b$, then transitivity implies that $y_{i+1} \le x_r$. But $y_{i+1} \le x_r$ is impossible because y_{i+1} and x_r are distinct (since $x_r \not\le y_{i+1}$) and have the same height. Hence every element in X is incomparable to every element in Y as claimed.

It follows that $X \cup Y$ induces a copy of $\underline{r} + \underline{j-i}$ in P. Because P is $(\underline{r} + \underline{s})$-free, we have that $j - i \le s - 1$ and therefore the set of integers $\{i+1, \ldots, j-1\}$ strictly between $I(x)$ and $I(y)$ has size at most $s-2$. $\square$

We now have the tools necessary to describe the initial t-society and our replacement scheme. While our transition rules require only that each S_j is a set of groups, our replacement scheme imposes additional structure on S_j. In particular, our replacement scheme treats S_j as a list of groups. Let q be the height of P. With I as in Lemma 3.1, we define $X_k = \{x \in P \colon\ k \in I(x)\}$ when $1 \le k \le q$ and set $S_0 = X_1, \ldots, X_q$. This ordering is preserved throughout the evolution: if Y appears before Z in S_0 and $\{Y, Z\} \subseteq S_j$, then Y also appears before Z in S_j. When L is a list of objects $a_1, \ldots, a_n$, we define $\mathrm{dist}_L(a_i, a_j) = |j - i|$. For convenience, when Y and Z are groups in S_j, we define $\mathrm{dist}_j(Y, Z) = \mathrm{dist}_{S_j}(Y, Z)$.

Let $t = 2(s-1)$. In the initial t-society (S_0, F_0), we define F_0 so that if Y and Z are distinct groups in S_0 with $\mathrm{dist}_0(Y, Z) \le s-1$, then $F(Y, k) = Z$ for some slot k. If fewer than $2(s-1)$ groups in S_0 are at distance at most $s-1$ from Y, then some slots are empty (formally, $F(Y, k) = \star$). Our replacement scheme maintains that in t-society (S_j, F_j), a group Y lists as friends all other groups Z such that $\mathrm{dist}_j(Y, Z) \le s-1$. This is possible to maintain since $\mathrm{dist}_j(Y, Z) < \mathrm{dist}_{j-1}(Y, Z)$ only occurs when some group $Z' \in S_{j-1}$ with $\mathrm{dist}_{j-1}(Y, Z') < \mathrm{dist}_{j-1}(Y, Z)$ does not survive the transition from (S_{j-1}, F_{j-1}) to (S_j, F_j). It follows that at least as many of Y's friendship slots become available as are needed to accommodate the groups Z with $\mathrm{dist}_{j-1}(Y, Z) > s-1$ and $\mathrm{dist}_j(Y, Z) \le s-1$. Our replacement scheme places these groups in Y's available friendship slots arbitrarily. As before, unused slots are assigned the value $\star$.

Our next aim is to show that our initial t-society and replacement scheme lead to a long evolution. We first prove an analogue of Lemma 4.2 in [17].

Lemma 3.2. *Let $C_1, \ldots, C_m$ be a First-Fit chain partition, and define $C_j = \varnothing$ for $j > m$. Let (S_0, F_0) be our initial t-society, and* *let $(S_0, F_0), \dots, (S_n, F_n)$ be the evolution resulting from our replacement scheme. For each i, we have that $\bigcup_{X \in S_i} X \supseteq \bigcup_{j>i} C_j$.*

Proof. By induction on i. By Lemma 3.1, $I(x) \neq \varnothing$ for each element x, and therefore $\bigcup_{X \in S_0} X$ contains all elements in P.

Let $i \geq 1$ and consider an element $y \in C_j$ with $j > i$. Because $C_1, \dots, C_m$ is a First-Fit chain partition, there is an element $z \in C_i$ such that y and z are incomparable. By induction, there are groups $Y \in S_{i-1}$ and $Z \in S_{i-1}$ with $y \in Y$ and $z \in Z$. Among all such pairs $\{Y, Z\}$, choose Y and Z to minimize $\mathrm{dist}_{i-1}(Y, Z)$. We claim that $\mathrm{dist}_{i-1}(Y, Z) \leq s - 1$. Indeed, if $\mathrm{dist}_{i-1}(Y, Z) \geq s$, then there are at least $s - 1$ groups in S_{i-1} that are strictly between Y and Z in the list $X_1, \dots, X_q$. By our selection of Y and Z, none of these groups contain y or z. Hence, it follows that the index of each such group is strictly between $I(y)$ and $I(z)$, contradicting Lemma 3.1.

Because $\mathrm{dist}_{i-1}(Y, Z) \leq s - 1$, our replacement scheme ensures that Y lists Z as a friend in some slot. Because $z \in Z \cap C_i$, some friend of Y in (S_{i-1}, F_{i-1}) has non-empty intersection with C_i. It follows that Y either makes an α-transition or a β-transition from S_{i-1} to S_i. Hence $y \in Y \in S_i$ and therefore $y \in \bigcup_{X \in S_i} X$ as required. □

Lemma 3.3. *Let $C_1, \dots, C_m$ be a First-Fit chain partition, and define $C_j = \varnothing$ for $j > m$. Let (S_0, F_0) be our initial t-society, and let $(S_0, F_0), \dots, (S_n, F_n)$ be the evolution resulting from our replacement scheme. We have that $n \geq m + 2$.*

Proof. Let $y \in C_m$. By Lemma 3.2, there is a group $Y \in S_{m-1}$ with $y \in Y$. Because Y has non-empty intersection with C_m, we have that Y makes an α-transition from S_{m-1} to S_m. Also, $N^{\alpha}_{m-1,m}(Y) = 1$ and $\varepsilon((m+1)-(m-1)) = 2\varepsilon = 1/t = 1/(2(s-1)) \leq 1/2$, and therefore Y is eligible to make a γ-transition from S_m to S_{m+1}. Hence $Y \in S_{m+1}$. Because the evolution ends with an empty t-society, it follows that $n \geq m + 2$. □

Putting all the pieces together, we obtain our main theorem.

Theorem 3.4. *If r and s are at least 2 and P is an $(\underline{r} + \underline{s})$-free poset of width w, then First-Fit partitions P into at most $8(r-1)(s-1)w$ chains.*

Proof. Let $C_1, \dots, C_m$ be a First-Fit chain partition, and define $C_j = \varnothing$ for $j > m$. Obtain our initial t-society (S_0, F_0) from Lemma 3.1, and let $(S_0, F_0), \dots, (S_n, F_n)$ be the evolution obtained with our replacement scheme. By Lemma 3.3, we have that $n \geq m + 2$. By Lemma 2.4, some group $X \in S_0$ has size at least $(n-2)/4t = (n-2)/(8(s-1)) \geq m/(8(s-1))$. By Lemma 3.1, the height of X is at most $r-1$. It follows that X is the union of $r-1$ antichains, and therefore $w \geq |X|/(r-1) \geq m/(8(s-1)(r-1))$. □

4. Concluding Remarks

The following related problem is open: for which posets Q of width 2 is there a function $f_Q(w)$ such that First-Fit partitions every Q-free poset of width w into at most $f_Q(w)$ chains? The same question applies when $f_Q(w)$ is restricted to be a polynomial or a linear function of w. We note that these problems are only interesting for posets Q of width 2. Indeed, there is a trivial linear bound when Q is a chain, and the example of Kierstead [9] implies that no such function exists when the width of Q is at least 3.

Addendum

While this article was under review, Bosek, Krawczyk, and Matecki [3] proved that for each poset Q of width 2, there is a function $f_Q(w)$ such that First-Fit partitions every Q-free poset of width w into at most $f_Q(w)$ chains. Our second question remains open.

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