First-Fit is Linear on $(\underline{r} + \underline{s})$-free Posets

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- (Szemerédi): $\mathrm{val}(w) \geq \binom{w+1}{2}$

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- (Bosek–Krawczyk (2010+)): $\mathrm{val}(w) \leq w^{16 \lg w}$
- (Bosek *et al.* (2010+)): $\mathrm{val}(w) \geq (2 - o(1))\binom{w+1}{2}$

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An interval order is a poset whose elements are closed intervals on the real line such that $[a, b] < [c, d]$ if and only if $b < c$.

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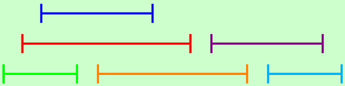
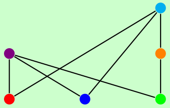


An Interval Order P



Hasse Diagram of P

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Theorem (Upper Bounds)

- (Woodall (1976)): $\mathrm{FF}(w) = O(w \log w)$
- (Kierstead (1988)): $\mathrm{FF}(w) \leq 40w$
- (Kierstead–Qin (1995)): $\mathrm{FF}(w) \leq 25.8w$
- (Pemmaraju–Raman–Varadarajan (2003)): $\mathrm{FF}(w) \leq 10w$
- (Brightwell–Kierstead–Trotter (2003; unpub)): $\mathrm{FF}(w) \leq 8w$
- (Narayansamy–Babu (2004)): $\mathrm{FF}(w) \leq 8w - 3$
- (Howard (2010+)): $\mathrm{FF}(w) \leq 8w - 4$

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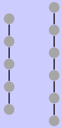
Theorem (Lower Bounds)

- (Kierstead–Trotter (1981)): There is a positive ε such that $\mathrm{FF}(w) \geq (3+\varepsilon)w$ when w is sufficiently large.
- (Chrobak–Ślusarek (1990)): $\mathrm{FF}(w) \geq 4w - 9$ when $w \geq 4$.
- (Kierstead–Trotter (2004)): $\mathrm{FF}(w) \geq 4.99w - O(1)$.
- (D. Smith (2009)): If $\varepsilon > 0$, then $\mathrm{FF}(w) \geq (5-\varepsilon)w$ when w is sufficiently large.

Beyond Interval Orders

Theorem (Fishburn (1970))



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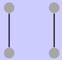

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- *The poset $\underline{r} + \underline{s}$ is the disjoint union of a chain of size r and a chain of size s.*
- *A poset P is an interval order if and only if P does not contain $\underline{2} + \underline{2}$ as an induced subposet.*

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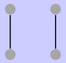

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If P is an $(\underline{r} + \underline{r})$-free poset of width w, then First-Fit partitions P into at most $3rw^2$ chains.

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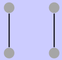

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Question (Bosek–Krawczyk–Szczypka (2010))

Can the bound be improved from $O(w^2)$ to $O(w)$?

Our Result

Theorem

If $r, s \geq 2$ *and* P *is an* $(\underline{r} + \underline{s})$*-free poset of width* w*, then First-Fit partitions* P *into at most* $8(r-1)(s-1)w$ *chains.*

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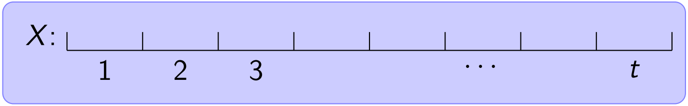

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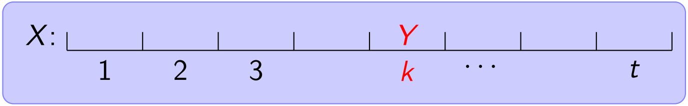

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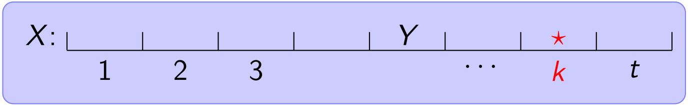

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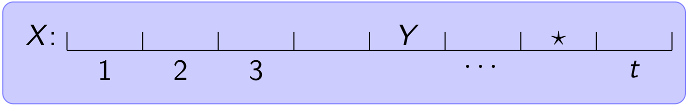

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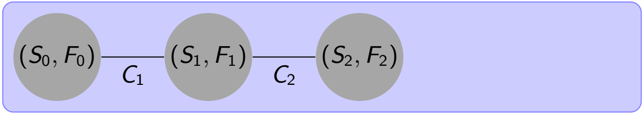

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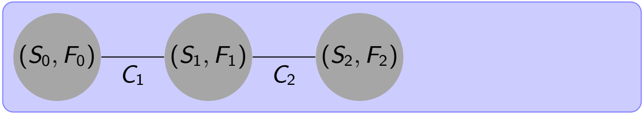

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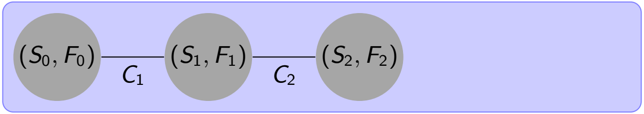

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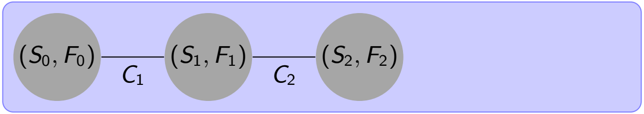

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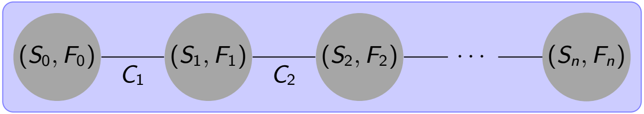

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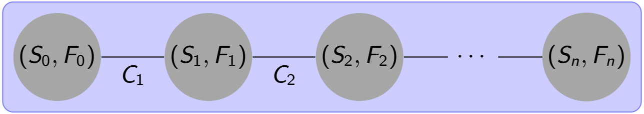

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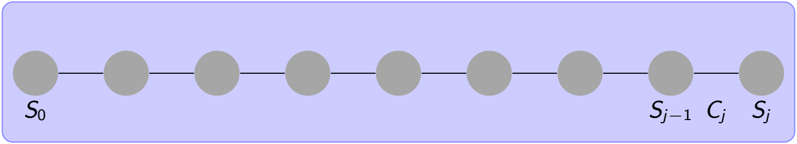
- The list $(S_0, F_0), \ldots, (S_n, F_n)$ is an evolution of societies.

Transition Rules



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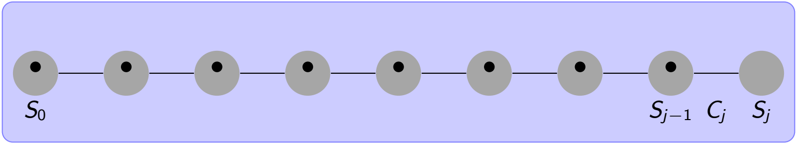

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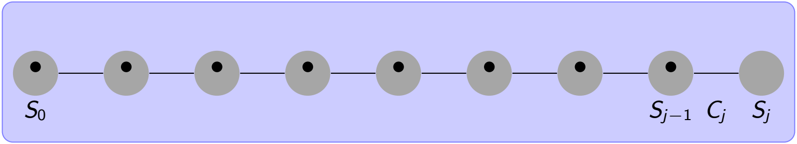

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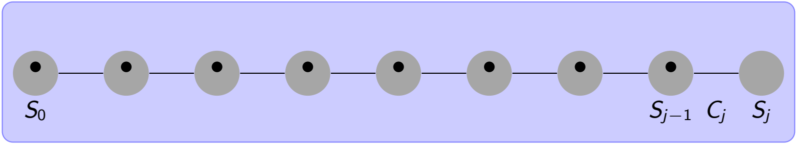

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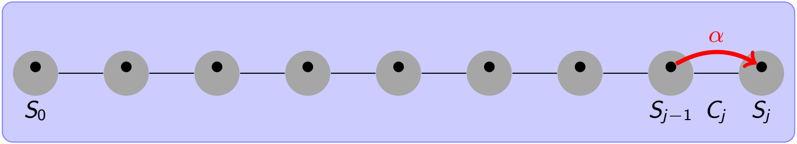

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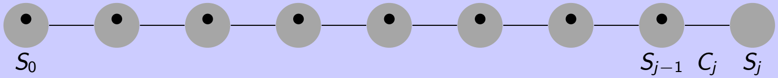

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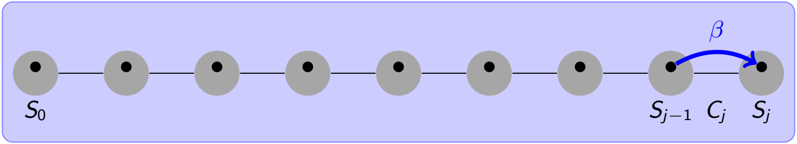

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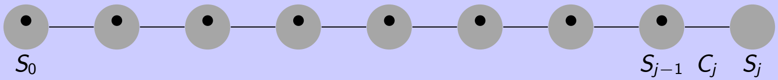

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3. Otherwise, if the number of α-transitions that X makes from S_i to S_{j-1} exceeds $(j-i)/(2t)$ for some i, then X makes a γ-transition to S_j.

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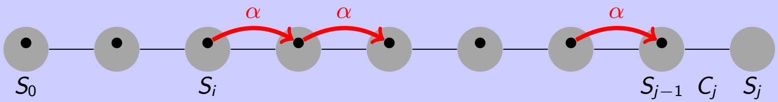

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2. Otherwise, if some friend of X in (S_{j-1}, F_{j-1}) has nonempty intersection with C_j, then X makes a β-transition to S_j.
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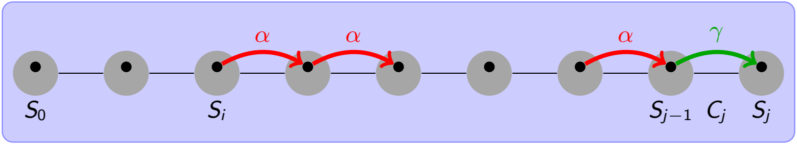

- Consider a group $X \in S_{j-1}$.
- There are 3 ways that X can transition from S_{j-1} to S_j.

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- Part 1 exploits that P is $(\underline{r} + \underline{s})$-free.
- Part 2 is essentially the standard analysis of the Column Construction Method of Pemmaraju, Raman, and Varadarajan.

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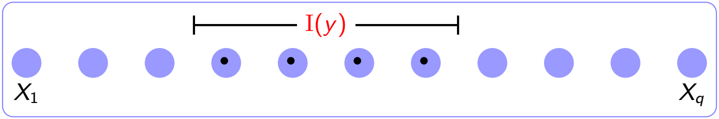

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- Define $S_0 = \{X_1, \ldots, X_q\}$.

Incomparable Elements are in Nearby Groups

Lemma

If y and z are incomparable, then either $\mathrm{I}(y) \cap \mathrm{I}(z) \neq \varnothing$, or there are at most $s - 2$ integers between $\mathrm{I}(y)$ and $\mathrm{I}(z)$.

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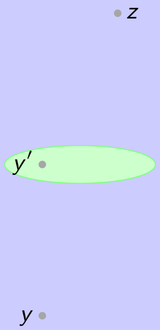

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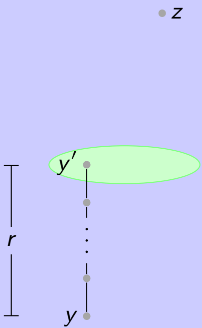

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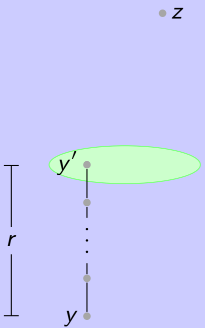

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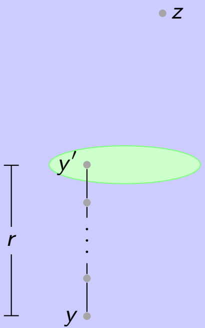

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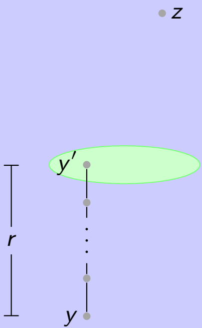

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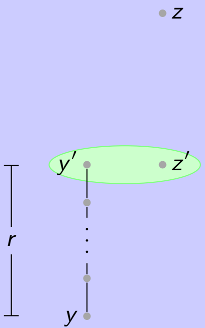

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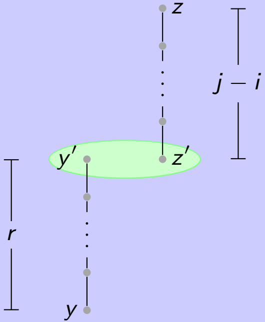

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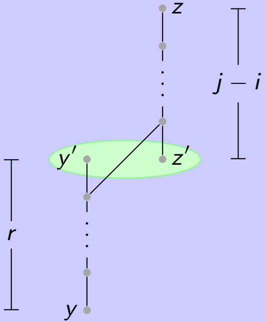

- If some element in Y is at most an element in Z,

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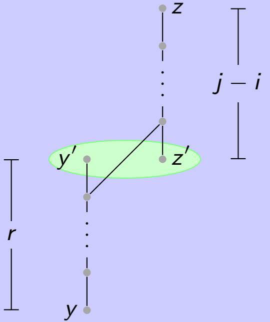

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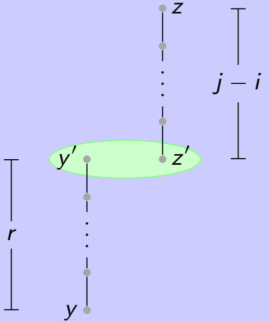

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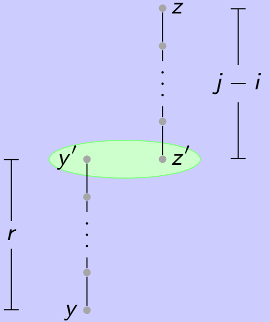

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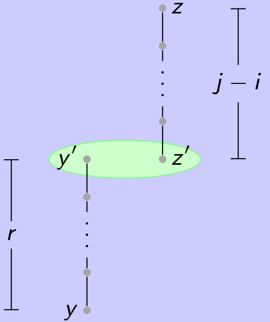

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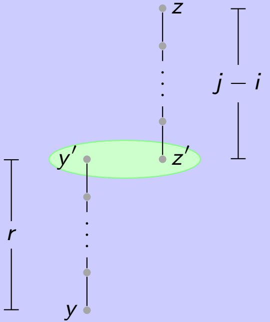

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- But y' and z' are distinct with the same adjusted height.

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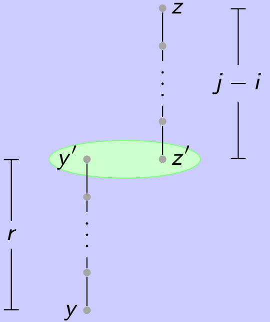

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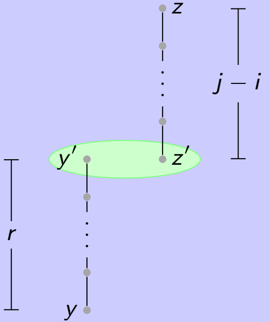

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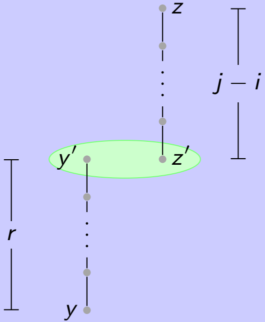

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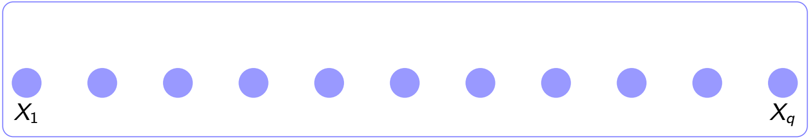
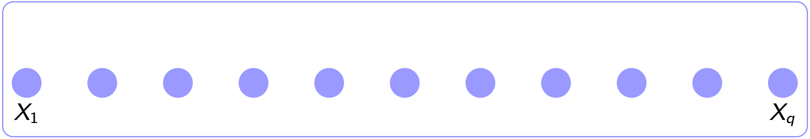
The Replacement Scheme



- Recall that each group has t slots for friends, where $t = 2(s-1)$.

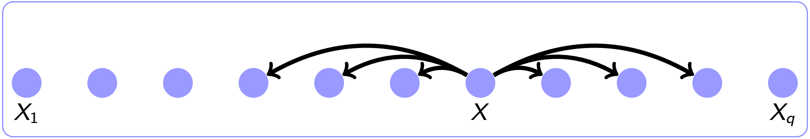
The Replacement Scheme



- Recall that each group has t slots for friends, where $t = 2(s-1)$.
- The replacement scheme maintains the invariant:

> The friends of X in (S_j, F_j) are the t groups closest to X among all that survive to S_j.

The Replacement Scheme



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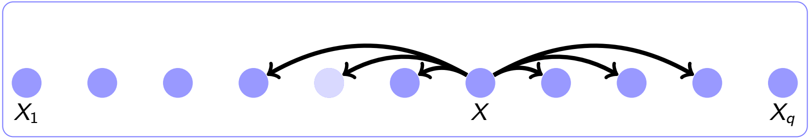

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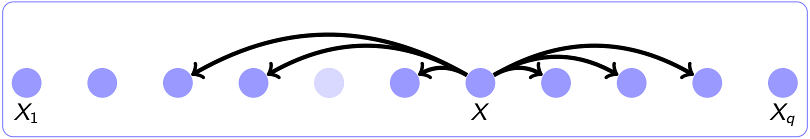

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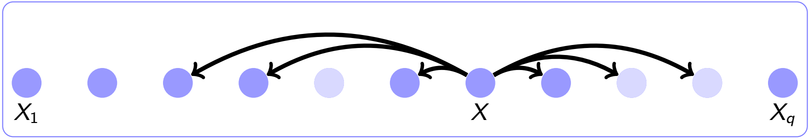

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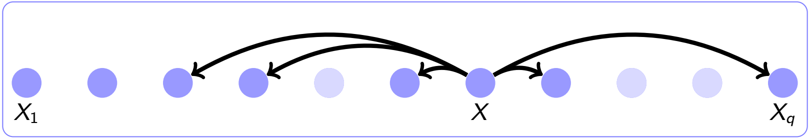

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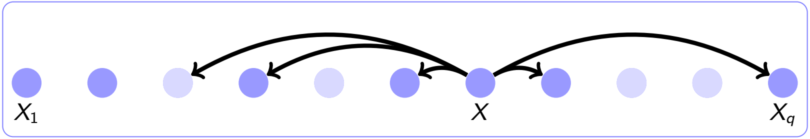

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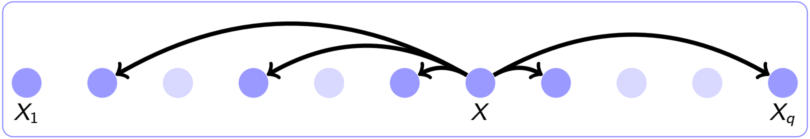

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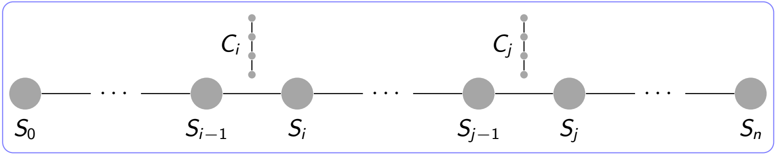
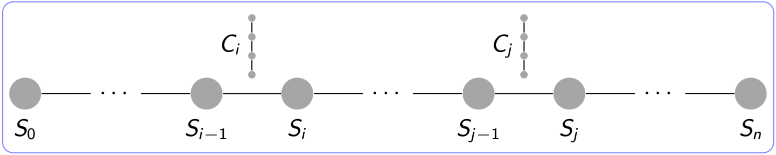
The Evolution is Long



Lemma

For each $i \geq 0$, we have that $\bigcup_{j>i} C_j \subseteq \bigcup_{X \in S_i} X$.

The Evolution is Long



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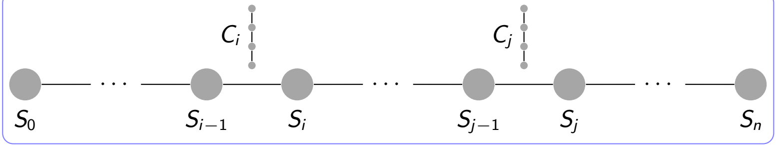
Proof.

- Induction on i.

□

The Evolution is Long



Lemma

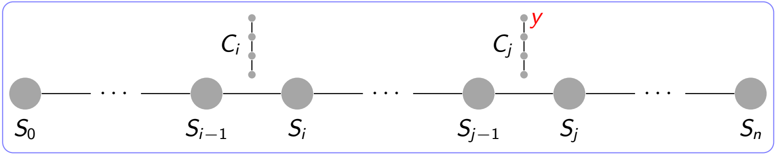
For each $i \geq 0$, we have that $\bigcup_{j>i} C_j \subseteq \bigcup_{X \in S_i} X$.

Proof.

- Induction on i.
- Case $i = 0$: each $y \in P$ is in a group in the initial society.

□

The Evolution is Long



Lemma

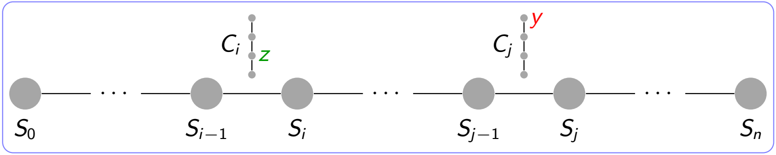
For each $i \geq 0$, we have that $\bigcup_{j>i} C_j \subseteq \bigcup_{X \in S_i} X$.

Proof.

- Suppose $i \geq 1$ and consider $y \in C_j$ where $j > i$.

□

The Evolution is Long



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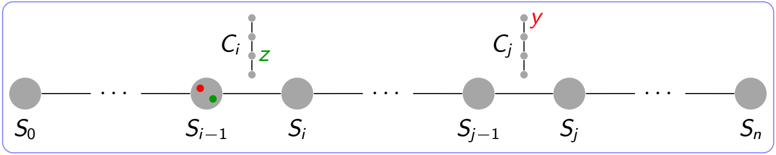
Proof.

- Suppose $i \geq 1$ and consider $y \in C_j$ where $j > i$.
- Find $z \in C_i$ such that y and z are incomparable.

□

The Evolution is Long



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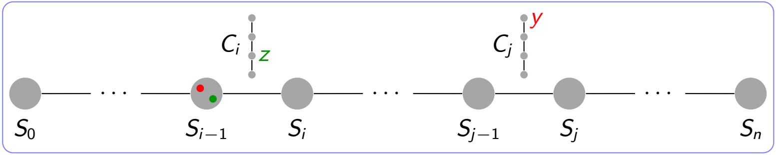
For each $i \geq 0$, we have that $\bigcup_{j>i} C_j \subseteq \bigcup_{X \in S_i} X$.

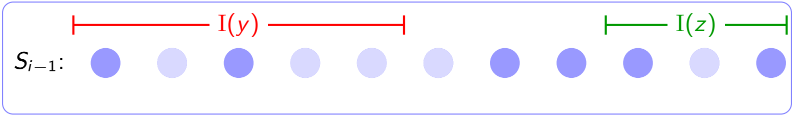
Proof.

- Suppose $i \geq 1$ and consider $y \in C_j$ where $j > i$.
- Find $z \in C_i$ such that y and z are incomparable.
- By induction, $\exists Y, Z \in S_{i-1}$ such that $y \in Y$ and $z \in Z$.

□

The Evolution is Long





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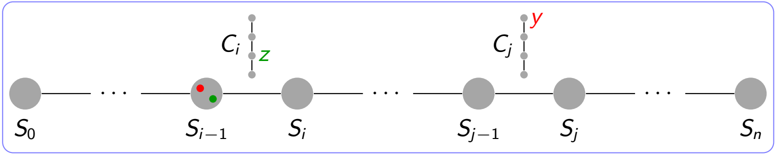
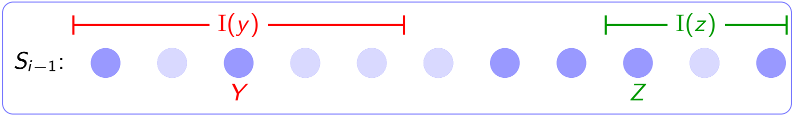
□

The Evolution is Long





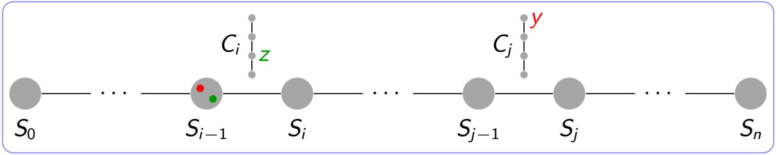
Lemma

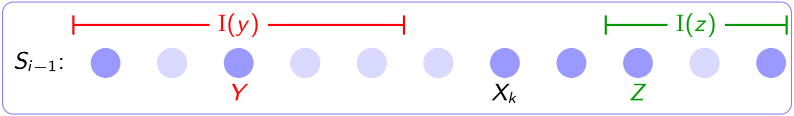
For each $i \geq 0$, we have that $\bigcup_{j>i} C_j \subseteq \bigcup_{X \in S_i} X$.

Proof.

- Suppose $i \geq 1$ and consider $y \in C_j$ where $j > i$.
- Find $z \in C_i$ such that y and z are incomparable.
- By induction, $\exists Y, Z \in S_{i-1}$ such that $y \in Y$ and $z \in Z$.
- Choose Y and Z as close as possible in $X_1, \ldots, X_q$. $\square$

The Evolution is Long





Lemma

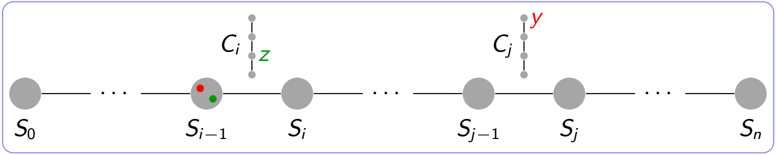
For each $i \geq 0$, we have that $\bigcup_{j>i} C_j \subseteq \bigcup_{X \in S_i} X$.

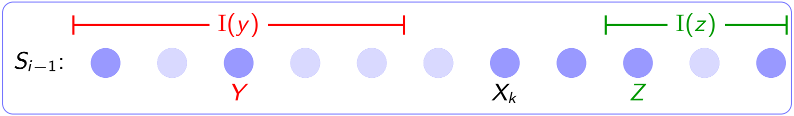
Proof.

- If X_k is a group that survives to S_{i-1} and is between Y and Z in $X_1, \ldots, X_q$, then k is between $\mathrm{I}(y)$ and $\mathrm{I}(z)$.

□

The Evolution is Long





Lemma

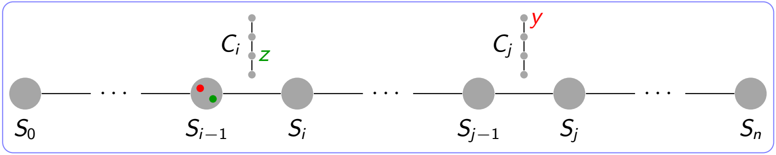
For each $i \geq 0$, we have that $\bigcup_{j>i} C_j \subseteq \bigcup_{X \in S_i} X$.

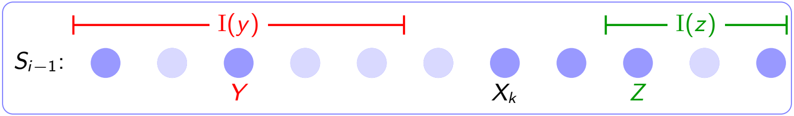
Proof.

- If X_k is a group that survives to S_{i-1} and is between Y and Z in $X_1, \ldots, X_q$, then k is between $\mathrm{I}(y)$ and $\mathrm{I}(z)$.
- Hence at most $s - 2$ groups in S_{i-1} are between Y and Z.

□

The Evolution is Long





Lemma

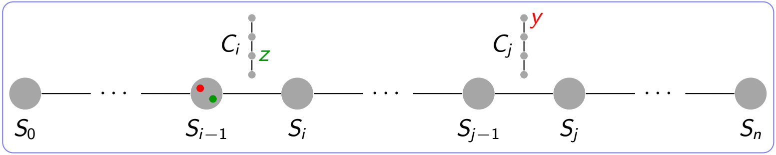
For each $i \geq 0$, we have that $\bigcup_{j>i} C_j \subseteq \bigcup_{X \in S_i} X$.

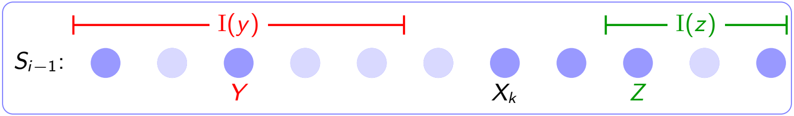
Proof.

- If $Y = Z$, then Y makes an α-transition to S_i.

□

The Evolution is Long





Lemma

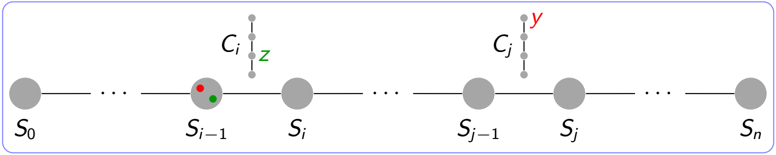
For each $i \geq 0$, we have that $\bigcup_{j>i} C_j \subseteq \bigcup_{X \in S_i} X$.

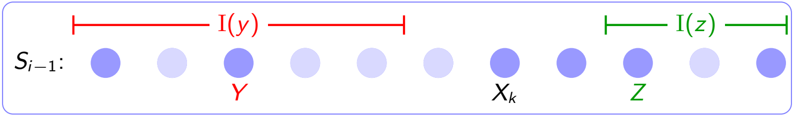
Proof.

- If $Y = Z$, then Y makes an α-transition to S_i.
- Otherwise, Y lists Z as a friend in (S_{i-1}, F_{i-1}).

□

The Evolution is Long





Lemma

For each $i \geq 0$, we have that $\bigcup_{j>i} C_j \subseteq \bigcup_{X \in S_i} X$.

Proof.

- If $Y = Z$, then Y makes an α-transition to S_i.
- Otherwise, Y lists Z as a friend in (S_{i-1}, F_{i-1}).
- Hence Y makes an α-transition or a β-transition to S_i.

□

Part 1 and Part 2

Lemma (Part 1)

If $C_1, \ldots, C_m$ *is a chain partition produced by First-Fit and* $(S_0, F_0), \ldots, (S_n, F_n)$ *is the resulting evolution, then* $n \geq m + 2$*.*

Part 1 and Part 2

Lemma (Part 1)

If $C_1, \ldots, C_m$ is a chain partition produced by First-Fit and $(S_0, F_0), \ldots, (S_n, F_n)$ is the resulting evolution, then $n \geq m + 2$.

Lemma (Part 2)

Let $C_1, \ldots, C_m$ be a chain partition produced by First-Fit and let $(S_0, F_0), \ldots, (S_n, F_n)$ be the resulting evolution. If $X \in S_{n-1}$, then $|X| \geq (n-2)/4t$.

Putting the Pieces Together

Theorem

If $r, s \geq 2$ and P is an $(\underline{r} + \underline{s})$-free poset of width w, then First-Fit partitions P into at most $8(r-1)(s-1)w$ chains.

Putting the Pieces Together

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If $r, s \geq 2$ and P is an $(\underline{r} + \underline{s})$-free poset of width w, then First-Fit partitions P into at most $8(r-1)(s-1)w$ chains.

Proof.

- Let $C_1, \ldots, C_m$ be a First-Fit chain partition.

□

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If $r, s \geq 2$ and P is an $(\underline{r} + \underline{s})$-free poset of width w, then First-Fit partitions P into at most $8(r-1)(s-1)w$ chains.

Proof.

- Let $C_1, \ldots, C_m$ be a First-Fit chain partition.
- Let $(S_0, F_0), \ldots, (S_n, F_n)$ be the resulting evolution.

$\square$

Putting the Pieces Together

Theorem

If $r, s \geq 2$ and P is an $(\underline{r} + \underline{s})$-free poset of width w, then First-Fit partitions P into at most $8(r-1)(s-1)w$ chains.

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- Let $C_1, \ldots, C_m$ be a First-Fit chain partition.
- Let $(S_0, F_0), \ldots, (S_n, F_n)$ be the resulting evolution.
- Let $X \in S_{n-1}$.

□

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If $r, s \geq 2$ and P is an $(\underline{r} + \underline{s})$-free poset of width w, then First-Fit partitions P into at most $8(r-1)(s-1)w$ chains.

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- Let $C_1, \ldots, C_m$ be a First-Fit chain partition.
- Let $(S_0, F_0), \ldots, (S_n, F_n)$ be the resulting evolution.
- Let $X \in S_{n-1}$.
- Since X has height at most $r - 1$, we have $w \geq |X|/(r-1)$.

□

Putting the Pieces Together

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If $r, s \geq 2$ and P is an $(\underline{r} + \underline{s})$-free poset of width w, then First-Fit partitions P into at most $8(r-1)(s-1)w$ chains.

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- Let $X \in S_{n-1}$.
- Since X has height at most $r-1$, we have $w \geq |X|/(r-1)$.
- By Part 2, $|X| \geq (n-2)/(4t)$, so $w \geq (n-2)/(4t(r-1))$.

□

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If $r, s \geq 2$ and P is an $(\underline{r} + \underline{s})$-free poset of width w, then First-Fit partitions P into at most $8(r-1)(s-1)w$ chains.

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- By Part 2, $|X| \geq (n-2)/(4t)$, so $w \geq (n-2)/(4t(r-1))$.
- By Part 1, $n \geq m+2$, so $w \geq m/(4t(r-1))$.

□

Putting the Pieces Together

Theorem

If $r, s \geq 2$ and P is an $(\underline{r} + \underline{s})$-free poset of width w, then First-Fit partitions P into at most $8(r-1)(s-1)w$ chains.

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- Let $C_1, \ldots, C_m$ be a First-Fit chain partition.
- Let $(S_0, F_0), \ldots, (S_n, F_n)$ be the resulting evolution.
- Let $X \in S_{n-1}$.
- Since X has height at most $r-1$, we have $w \geq |X|/(r-1)$.
- By Part 2, $|X| \geq (n-2)/(4t)$, so $w \geq (n-2)/(4t(r-1))$.
- By Part 1, $n \geq m+2$, so $w \geq m/(4t(r-1))$.
- Since $t = 2(s-1)$, we have $w \geq m/(8(s-1)(r-1))$.

□

Open Problems

Theorem

If $r, s \geq 2$ and P is an $(\underline{r} + \underline{s})$-free poset of width w, then First-Fit partitions P into at most $8(r-1)(s-1)w$ chains.

Open Problems

Theorem

If $r, s \geq 2$ and P is an $(\underline{r} + \underline{s})$-free poset of width w, then First-Fit partitions P into at most $8(r-1)(s-1)w$ chains.

- Improve the constant $8(r-1)(s-1)$ in the upper bound.

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- Improve the constant $8(r-1)(s-1)$ in the upper bound.
- Give lower bounds when $(r, s) \neq (2, 2)$.

Open Problems

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If $r, s \geq 2$ and P is an $(\underline{r} + \underline{s})$-free poset of width w, then First-Fit partitions P into at most $8(r-1)(s-1)w$ chains.

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Question

For which posets Q is there a function f such that First-Fit partitions a Q-free poset of width w into at most $f(w)$ chains?

Open Problems

Theorem

If $r, s \geq 2$ and P is an $(\underline{r} + \underline{s})$-free poset of width w, then First-Fit partitions P into at most $8(r-1)(s-1)w$ chains.

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Question

For which posets Q is there a function f such that First-Fit partitions a Q-free poset of width w into at most $f(w)$ chains?

- Note: Kierstead's example shows that Q must have width 2.