Turán and Ramsey Results for Boolean Algebras

Kevin G. Milans (milans@math.wvu.edu)
Joint with L. Lu and J. T. Johnston

West Virginia University

AMS Spring Central Section Meeting
Iowa State University
28 April 2013

Boolean Algebras

- Let $[n] = \{1, \ldots, n\}$.

Boolean Algebras

- Let $[n] = \{1, \ldots, n\}$.
- The *n*-dimensional Boolean lattice is the containment order on $2^{[n]}$.

Boolean Algebras

- Let $[n] = \{1, \ldots, n\}$.
- The *n-dimensional Boolean lattice* is the containment order on $2^{[n]}$.

Definition

- Given disjoint sets $X_0, X_1, \ldots, X_d$, with $X_i \neq \emptyset$ for $i \geq 1$,

Boolean Algebras

- Let $[n] = \{1, \ldots, n\}$.
- The n-dimensional Boolean lattice is the containment order on $2^{[n]}$.

Definition

- Given disjoint sets $X_0, X_1, \ldots, X_d$, with $X_i \neq \varnothing$ for $i \geq 1$, the generated d-dimensional Boolean algebra is the family of all sets formed by the union of X_0 with 0 or more members of $\{X_1, \ldots, X_d\}$.

Boolean Algebras

- Let $[n] = \{1, \ldots, n\}$.
- The *n*-dimensional Boolean lattice is the containment order on $2^{[n]}$.

Definition

- Given disjoint sets $X_0, X_1, \ldots, X_d$, with $X_i \neq \varnothing$ for $i \geq 1$, the generated *d*-dimensional Boolean algebra is the family of all sets formed by the union of X_0 with 0 or more members of $\{X_1, \ldots, X_d\}$.
- Such a family of 2^d sets forms a copy of $\mathcal{B}_d$.

Boolean Algebras

- Let $[n] = \{1, \ldots, n\}$.
- The *n-dimensional Boolean lattice* is the containment order on $2^{[n]}$.

Definition

- Given disjoint sets $X_0, X_1, \ldots, X_d$, with $X_i \neq \varnothing$ for $i \geq 1$, the *generated d-dimensional Boolean algebra* is the family of all sets formed by the union of X_0 with 0 or more members of $\{X_1, \ldots, X_d\}$.
- Such a family of 2^d sets forms a *copy of $\mathcal{B}_d$*.
- A family is *$\mathcal{B}_d$-free* if it does not contain a copy of $\mathcal{B}_d$.

Turán Problem

- What is the largest size of a $\mathcal{B}_d$-free subfamily of $2^{[n]}$?

Turán Problem

- What is the largest size of a $\mathcal{B}_d$-free subfamily of $2^{[n]}$?
- Let $b(n,d) = \max\left\{|\mathcal{F}| : \mathcal{F} \subseteq 2^{[n]} \text{ and } \mathcal{F} \text{ is } \mathcal{B}_d\text{-free}\right\}$.

Turán Problem

- What is the largest size of a $\mathcal{B}_d$-free subfamily of $2^{[n]}$?
- Let $b(n,d) = \max\left\{|\mathcal{F}| : \mathcal{F} \subseteq 2^{[n]} \text{ and } \mathcal{F} \text{ is } \mathcal{B}_d\text{-free}\right\}$.

Prior Work

- [Sperner] $b(n,1) = \binom{n}{\lfloor n/2 \rfloor} \sim \sqrt{2/\pi} \cdot n^{-1/2} \cdot 2^n$

Turán Problem

- What is the largest size of a $\mathcal{B}_d$-free subfamily of $2^{[n]}$?
- Let $b(n,d) = \max\left\{|\mathcal{F}| : \mathcal{F} \subseteq 2^{[n]} \text{ and } \mathcal{F} \text{ is } \mathcal{B}_d\text{-free}\right\}$.

Prior Work

- [Sperner] $b(n,1) = \binom{n}{\lfloor n/2 \rfloor} \sim \sqrt{2/\pi} \cdot n^{-1/2} \cdot 2^n$
- [Erdős–Kleitman 1971] For some constants c_1, c_2 and n sufficiently large

$$c_1 \cdot n^{-1/4} \cdot 2^n \leq b(n,2) \leq c_2 \cdot n^{-1/4} \cdot 2^n.$$

Turán Problem

- What is the largest size of a $\mathcal{B}_d$-free subfamily of $2^{[n]}$?
- Let $b(n, d) = \max\left\{|\mathcal{F}| : \mathcal{F} \subseteq 2^{[n]} \text{ and } \mathcal{F} \text{ is } \mathcal{B}_d\text{-free}\right\}$.

Prior Work

- [Gunderson–Rödl–Sidorenko 1999] For each d, there exists c_d such that for n sufficiently large

$$n^{-\frac{d}{2^{d+1}-2}(1-o(1))} \cdot 2^n \leq b(n, d) \leq c_d \cdot n^{-\frac{1}{2^d}} \cdot 2^n.$$

Turán Problem

- What is the largest size of a $\mathcal{B}_d$-free subfamily of $2^{[n]}$?
- Let $b(n,d) = \max\left\{|\mathcal{F}| : \mathcal{F} \subseteq 2^{[n]} \text{ and } \mathcal{F} \text{ is } \mathcal{B}_d\text{-free}\right\}$.

Prior Work

- [Gunderson–Rödl–Sidorenko 1999] For each d, there exists c_d such that for n sufficiently large

$$n^{-\frac{d}{2^{d+1}-2}(1-o(1))} \cdot 2^n \leq b(n,d) \leq c_d \cdot n^{-\frac{1}{2^d}} \cdot 2^n.$$

- Here, $c_d = (10d)^d(1+o(1))$.

Turán Problem

- What is the largest size of a $\mathcal{B}_d$-free subfamily of $2^{[n]}$?
- Let $b(n,d) = \max\left\{|\mathcal{F}| : \mathcal{F} \subseteq 2^{[n]} \text{ and } \mathcal{F} \text{ is } \mathcal{B}_d\text{-free}\right\}$.

Prior Work

- [Gunderson–Rödl–Sidorenko 1999] For each d, there exists c_d such that for n sufficiently large

$$n^{-\frac{d}{2^{d+1}-2}(1-o(1))} \cdot 2^n \leq b(n,d) \leq c_d \cdot n^{-\frac{1}{2^d}} \cdot 2^n.$$

- Here, $c_d = (10d)^d(1+o(1))$.

Theorem

$$b(n,d) \leq 50 \cdot n^{-\frac{1}{2^d}} \cdot 2^n.$$

A useful sequence of functions

Definition

- Let $\alpha_0(n) = 0$.

A useful sequence of functions

Definition

- Let $\alpha_0(n) = 0$.
- For $d \geq 1$, define $\alpha_d(n) = \frac{1}{2} + \sqrt{2n\alpha_{d-1}(n) + \frac{1}{4}}$.

A useful sequence of functions

Definition

- Let $\alpha_0(n) = 0$.
- For $d \geq 1$, define $\alpha_d(n) = \frac{1}{2} + \sqrt{2n\alpha_{d-1}(n) + \frac{1}{4}}$.

Facts

- For $d \geq 1$, the bounds $(2n)^{1-\frac{2}{2^d}} \leq \alpha_d(n) \leq (4n)^{1-\frac{2}{2^d}}$ hold.

A useful sequence of functions

Definition

- Let $\alpha_0(n) = 0$.
- For $d \geq 1$, define $\alpha_d(n) = \frac{1}{2} + \sqrt{2n\alpha_{d-1}(n) + \frac{1}{4}}$.

Facts

- For $d \geq 1$, the bounds $(2n)^{1-\frac{2}{2^d}} \leq \alpha_d(n) \leq (4n)^{1-\frac{2}{2^d}}$ hold.
- For fixed d, we have $\alpha_d(n) = (1 + o(1))(2n)^{1-\frac{2}{2^d}}$.

A useful sequence of functions

Definition

- Let $\alpha_0(n) = 0$.
- For $d \geq 1$, define $\alpha_d(n) = \frac{1}{2} + \sqrt{2n\alpha_{d-1}(n) + \frac{1}{4}}$.

Facts

- For $d \geq 1$, the bounds $(2n)^{1-\frac{2}{2^d}} \leq \alpha_d(n) \leq (4n)^{1-\frac{2}{2^d}}$ hold.
- For fixed d, we have $\alpha_d(n) = (1+o(1))(2n)^{1-\frac{2}{2^d}}$.
- For $d \geq 1$, we have $\binom{\alpha_d(n)}{2}/n = \alpha_{d-1}(n)$.

Szemerédi's Cube Lemma

- Given $x_0, x_1, \ldots, x_d$ with $x_0 \geq 0$ and $x_i \geq 1$ for $i \geq 1$,

Szemerédi's Cube Lemma

- Given $x_0, x_1, \ldots, x_d$ with $x_0 \geq 0$ and $x_i \geq 1$ for $i \geq 1$, the generated affine d-cube is the set of all integers obtained by adding x_0 to the sum of 0 or more members of $\{x_1, \ldots, x_d\}$.

Szemerédi's Cube Lemma

- Given $x_0, x_1, \ldots, x_d$ with $x_0 \geq 0$ and $x_i \geq 1$ for $i \geq 1$, the generated affine d-cube is the set of all integers obtained by adding x_0 to the sum of 0 or more members of $\{x_1, \ldots, x_d\}$.

Lemma (Szemerédi 1969)

If $A \subseteq [0, n]$ and $|A| > \alpha_d(n)$, then A contains an affine d-cube.

Szemerédi's Cube Lemma

- Given $x_0, x_1, \ldots, x_d$ with $x_0 \geq 0$ and $x_i \geq 1$ for $i \geq 1$, the generated affine d-cube is the set of all integers obtained by adding x_0 to the sum of 0 or more members of $\{x_1, \ldots, x_d\}$.

Lemma (Szemerédi 1969)

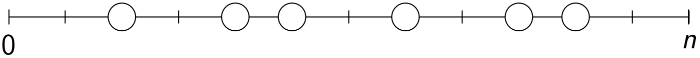
If $A \subseteq [0, n]$ and $|A| > \alpha_d(n)$, then A contains an affine d-cube.

Proof.

- By induction on d; case $d = 0$ is easy. Let $d \geq 1$ and $A \subseteq [0, n]$ with $|A| > \alpha_d(n)$.

□

Szemerédi's Cube Lemma



Lemma (Szemerédi 1969)

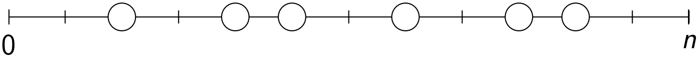
If $A \subseteq [0, n]$ *and* $|A| > \alpha_d(n)$*, then* A *contains an affine* d*-cube.*

Proof.

- By induction on d; case $d = 0$ is easy. Let $d \geq 1$ and $A \subseteq [0, n]$ with $|A| > \alpha_d(n)$.

□

Szemerédi's Cube Lemma



Lemma (Szemerédi 1969)

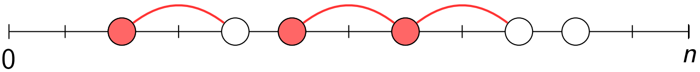
If $A \subseteq [0, n]$ and $|A| > \alpha_d(n)$, then A contains an affine d-cube.

Proof.

- By induction on d; case $d = 0$ is easy. Let $d \geq 1$ and $A \subseteq [0, n]$ with $|A| > \alpha_d(n)$.
- For each $k \in [n]$, let $A_k = \{a \in A\colon\ a + k \in A\}$.

□

Szemerédi's Cube Lemma



Lemma (Szemerédi 1969)

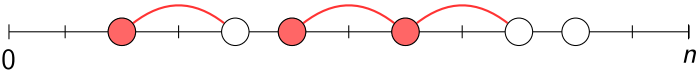
If $A \subseteq [0, n]$ and $|A| > \alpha_d(n)$, then A contains an affine d-cube.

Proof.

- By induction on d; case $d = 0$ is easy. Let $d \geq 1$ and $A \subseteq [0, n]$ with $|A| > \alpha_d(n)$.
- For each $k \in [n]$, let $A_k = \{a \in A\colon\ a + k \in A\}$.

□

Szemerédi's Cube Lemma



Lemma (Szemerédi 1969)

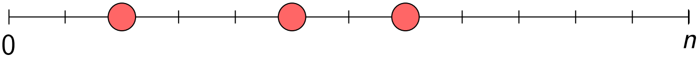
If $A \subseteq [0, n]$ and $|A| > \alpha_d(n)$, then A contains an affine d-cube.

Proof.

- By induction on d; case $d = 0$ is easy. Let $d \geq 1$ and $A \subseteq [0, n]$ with $|A| > \alpha_d(n)$.
- For each $k \in [n]$, let $A_k = \{a \in A\colon\ a + k \in A\}$.
- $\sum_{k=1}^{n} |A_k| = \binom{|A|}{2}$.

□

Szemerédi's Cube Lemma



Lemma (Szemerédi 1969)

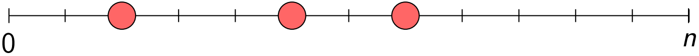
If $A \subseteq [0,n]$ and $|A| > \alpha_d(n)$, then A contains an affine d-cube.

Proof.

- By induction on d; case $d = 0$ is easy. Let $d \geq 1$ and $A \subseteq [0,n]$ with $|A| > \alpha_d(n)$.
- For each $k \in [n]$, let $A_k = \{a \in A\colon\ a + k \in A\}$.
- $\sum_{k=1}^{n} |A_k| = \binom{|A|}{2}$. Find k with $|A_k| \geq \binom{|A|}{2}/n > \alpha_{d-1}(n)$.

□

Szemerédi's Cube Lemma



Lemma (Szemerédi 1969)

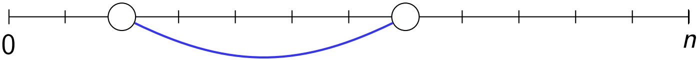
If $A \subseteq [0, n]$ and $|A| > \alpha_d(n)$, then A contains an affine d-cube.

Proof.

- By induction on d; case $d = 0$ is easy. Let $d \geq 1$ and $A \subseteq [0, n]$ with $|A| > \alpha_d(n)$.
- For each $k \in [n]$, let $A_k = \{a \in A\colon\ a + k \in A\}$.
- $\sum_{k=1}^{n} |A_k| = \binom{|A|}{2}$. Find k with $|A_k| \geq \binom{|A|}{2}/n > \alpha_{d-1}(n)$.
- Induction: A_k has an affine cube generated by $x_0, \ldots, x_{d-1}$.

□

Szemerédi's Cube Lemma



Lemma (Szemerédi 1969)

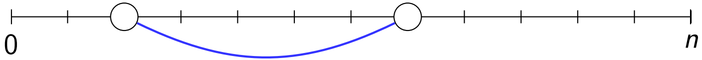
If $A \subseteq [0, n]$ and $|A| > \alpha_d(n)$, then A contains an affine d-cube.

Proof.

- By induction on d; case $d = 0$ is easy. Let $d \geq 1$ and $A \subseteq [0, n]$ with $|A| > \alpha_d(n)$.
- For each $k \in [n]$, let $A_k = \{a \in A\colon\ a + k \in A\}$.
- $\sum_{k=1}^{n} |A_k| = \binom{|A|}{2}$. Find k with $|A_k| \geq \binom{|A|}{2}/n > \alpha_{d-1}(n)$.
- Induction: A_k has an affine cube generated by $x_0, \ldots, x_{d-1}$.

□

Szemerédi's Cube Lemma



Lemma (Szemerédi 1969)

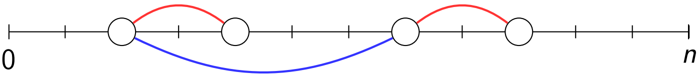
If $A \subseteq [0, n]$ and $|A| > \alpha_d(n)$, then A contains an affine d-cube.

Proof.

- By induction on d; case $d = 0$ is easy. Let $d \geq 1$ and $A \subseteq [0, n]$ with $|A| > \alpha_d(n)$.
- For each $k \in [n]$, let $A_k = \{a \in A\colon\ a + k \in A\}$.
- $\sum_{k=1}^{n} |A_k| = \binom{|A|}{2}$. Find k with $|A_k| \geq \binom{|A|}{2}/n > \alpha_{d-1}(n)$.
- Induction: A_k has an affine cube generated by $x_0, \ldots, x_{d-1}$.
- A has an affine cube generated by $x_0, \ldots, x_d$ with $x_d = k$.

□

Szemerédi's Cube Lemma



Lemma (Szemerédi 1969)

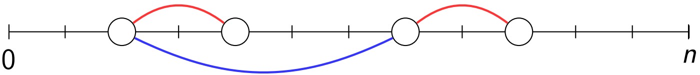
If $A \subseteq [0, n]$ and $|A| > \alpha_d(n)$, then A contains an affine d-cube.

Proof.

- By induction on d; case $d = 0$ is easy. Let $d \geq 1$ and $A \subseteq [0, n]$ with $|A| > \alpha_d(n)$.
- For each $k \in [n]$, let $A_k = \{a \in A \colon a + k \in A\}$.
- $\sum_{k=1}^{n} |A_k| = \binom{|A|}{2}$. Find k with $|A_k| \geq \binom{|A|}{2}/n > \alpha_{d-1}(n)$.
- Induction: A_k has an affine cube generated by $x_0, \ldots, x_{d-1}$.
- A has an affine cube generated by $x_0, \ldots, x_d$ with $x_d = k$.

□

Szemerédi's Cube Lemma



Lemma (Szemerédi 1969)

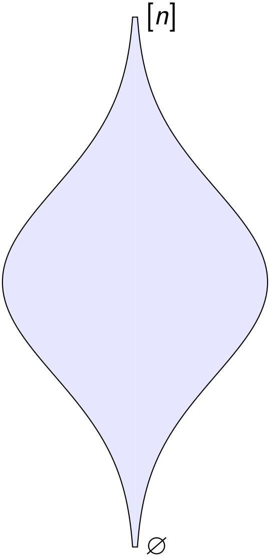
If $A \subseteq [0, n]$ *and* $|A| > \alpha_d(n)$, *then* A *contains an affine* d*-cube.*

- Using $\alpha_d(n) \leq (4n)^{1-\frac{2}{2^d}} < 4n^{1-\frac{2}{2^d}}$, we obtain:

Corollary

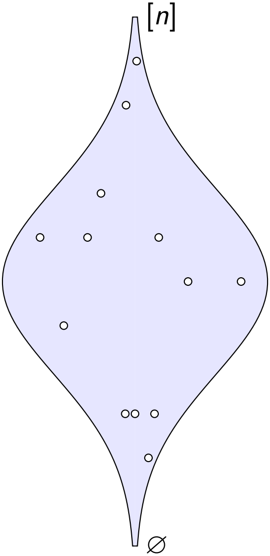
If $A \subseteq [0, n]$ *and* $|A| \geq 4n^{1-\frac{2}{2^d}}$, *then* A *contains an affine* d*-cube.*

The Lubell Function



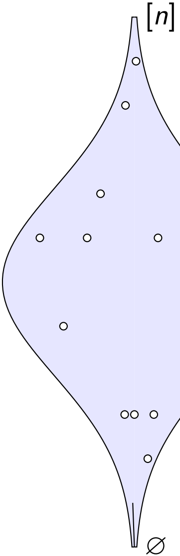
- Given $\mathcal{F} \subseteq 2^{[n]}$, let X be the number of times a random full chain meets $\mathcal{F}$.

The Lubell Function



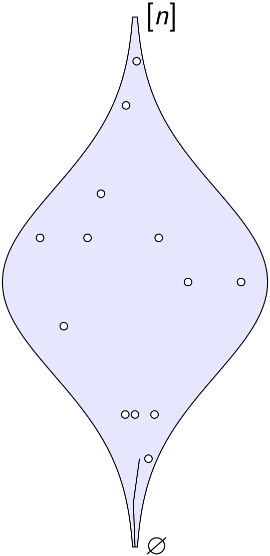
- Given $\mathcal{F} \subseteq 2^{[n]}$, let X be the number of times a random full chain meets $\mathcal{F}$.

The Lubell Function



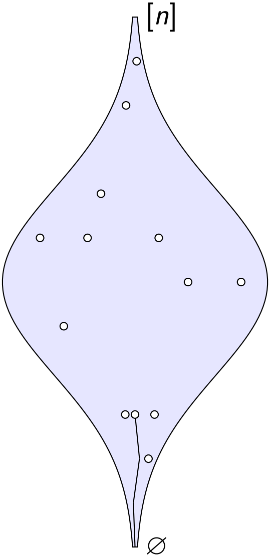
- Given $\mathcal{F} \subseteq 2^{[n]}$, let X be the number of times a random full chain meets $\mathcal{F}$.

The Lubell Function



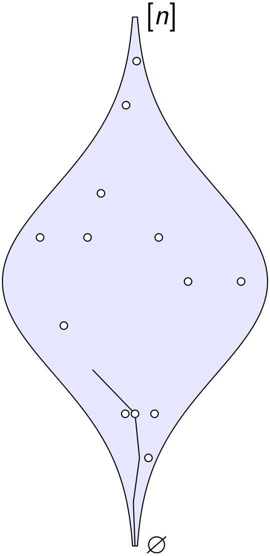
- Given $\mathcal{F} \subseteq 2^{[n]}$, let X be the number of times a random full chain meets $\mathcal{F}$.

The Lubell Function



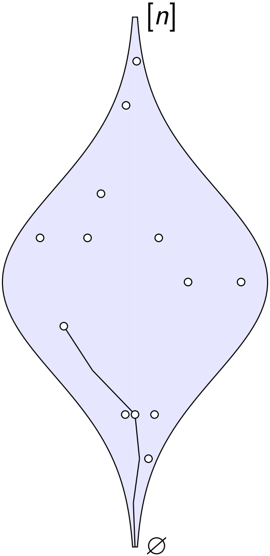
- Given $\mathcal{F} \subseteq 2^{[n]}$, let X be the number of times a random full chain meets $\mathcal{F}$.

The Lubell Function



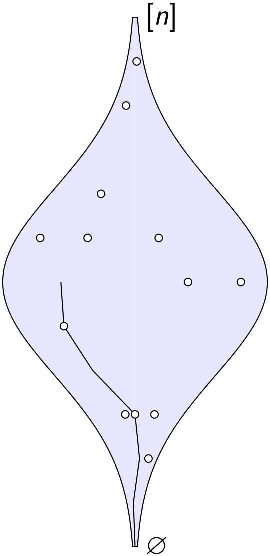
- Given $\mathcal{F} \subseteq 2^{[n]}$, let X be the number of times a random full chain meets $\mathcal{F}$.

The Lubell Function



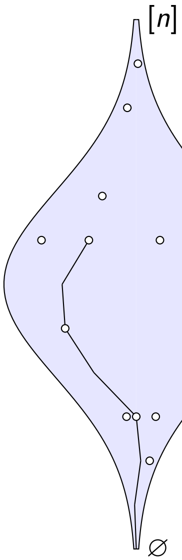
- Given $\mathcal{F} \subseteq 2^{[n]}$, let X be the number of times a random full chain meets $\mathcal{F}$.

The Lubell Function



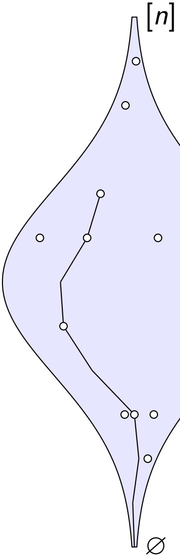
- Given $\mathcal{F} \subseteq 2^{[n]}$, let X be the number of times a random full chain meets $\mathcal{F}$.

The Lubell Function



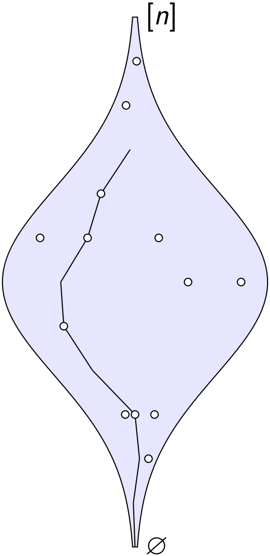
- Given $\mathcal{F} \subseteq 2^{[n]}$, let X be the number of times a random full chain meets $\mathcal{F}$.

The Lubell Function



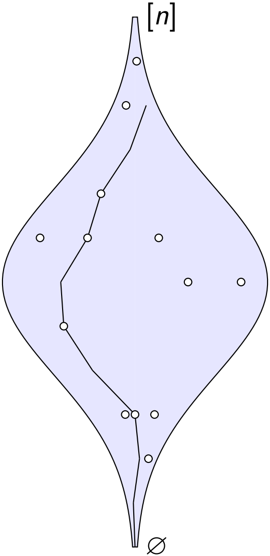
- Given $\mathcal{F} \subseteq 2^{[n]}$, let X be the number of times a random full chain meets $\mathcal{F}$.

The Lubell Function



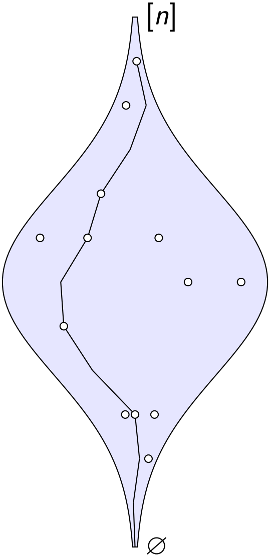
- Given $\mathcal{F} \subseteq 2^{[n]}$, let X be the number of times a random full chain meets $\mathcal{F}$.

The Lubell Function



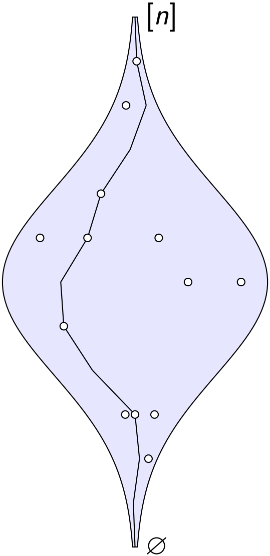
- Given $\mathcal{F} \subseteq 2^{[n]}$, let X be the number of times a random full chain meets $\mathcal{F}$.

The Lubell Function



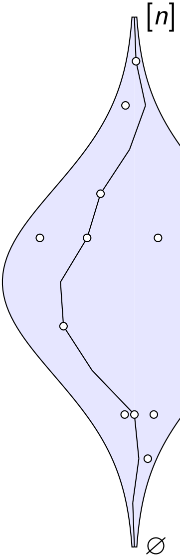
- Given $\mathcal{F} \subseteq 2^{[n]}$, let X be the number of times a random full chain meets $\mathcal{F}$.

The Lubell Function



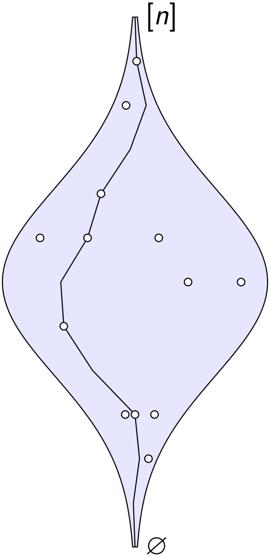
- Given $\mathcal{F} \subseteq 2^{[n]}$, let X be the number of times a random full chain meets $\mathcal{F}$.

The Lubell Function



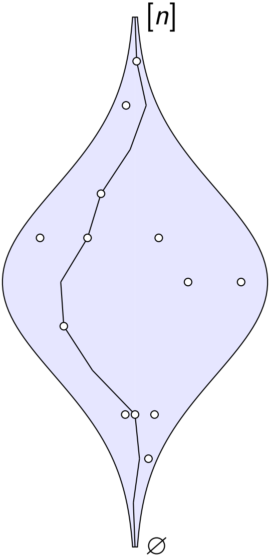
- Given $\mathcal{F} \subseteq 2^{[n]}$, let X be the number of times a random full chain meets $\mathcal{F}$.
- $\mathbf{E}[X] = \sum_{A \in \mathcal{F}} \frac{1}{\binom{n}{|A|}}$

The Lubell Function



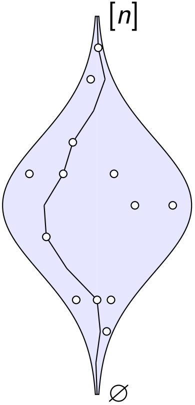
- Given $\mathcal{F} \subseteq 2^{[n]}$, let X be the number of times a random full chain meets $\mathcal{F}$.
- $\mathbf{E}[X] = \sum_{A \in \mathcal{F}} \frac{1}{\binom{n}{|A|}}$
- The Lubell function of $\mathcal{F}$, denoted $h_n(\mathcal{F})$, is $\mathbf{E}[X]$.

The Lubell Function



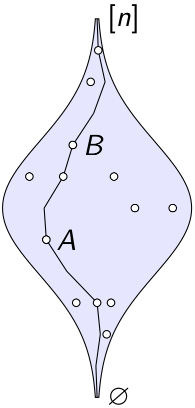
- Given $\mathcal{F} \subseteq 2^{[n]}$, let X be the number of times a random full chain meets $\mathcal{F}$.
- $\mathbf{E}[X] = \sum_{A \in \mathcal{F}} \frac{1}{\binom{n}{|A|}}$
- The Lubell function of $\mathcal{F}$, denoted $h_n(\mathcal{F})$, is $\mathbf{E}[X]$.
- Think of $h_n(\mathcal{F})$ as a measure of the size of $\mathcal{F}$, with $0 \leq h_n(\mathcal{F}) \leq n + 1$.

The Second Moment



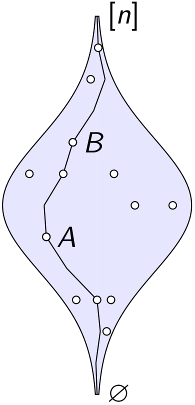
- $\binom{X}{2}$ also gives useful information.

The Second Moment



- $\binom{X}{2}$ also gives useful information.
- For each ordered pair (A, B) of distinct elements in $\mathcal{F}$ with $A \subsetneq B$, let $Y_{A,B}$ be the indicator r.v. for the full chain containing A and B.

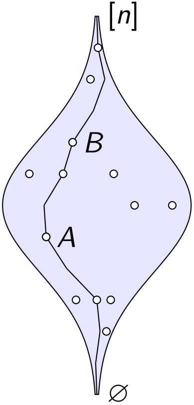
The Second Moment



- $\binom{X}{2}$ also gives useful information.
- For each ordered pair (A, B) of distinct elements in $\mathcal{F}$ with $A \subsetneq B$, let $Y_{A,B}$ be the indicator r.v. for the full chain containing A and B.

$$\mathbf{E}[\tbinom{X}{2}] = \sum_{A,B} \mathbf{E}\left[Y_{A,B}\right]$$

The Second Moment

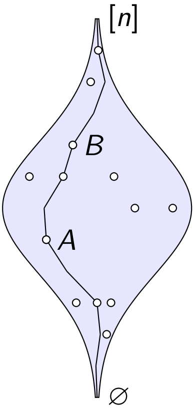


- $\binom{X}{2}$ also gives useful information.
- For each ordered pair (A, B) of distinct elements in $\mathcal{F}$ with $A \subsetneq B$, let $Y_{A,B}$ be the indicator r.v. for the full chain containing A and B.

$$\mathbf{E}[\tbinom{X}{2}] = \sum_{A,B} \mathbf{E}\left[Y_{A,B}\right]$$

$$= \sum_{A,B} \frac{1}{\binom{n}{|A|,|B|-|A|,n-|B|}}$$

The Second Moment



- $\binom{X}{2}$ also gives useful information.
- For each ordered pair (A, B) of distinct elements in $\mathcal{F}$ with $A \subsetneq B$, let $Y_{A,B}$ be the indicator r.v. for the full chain containing A and B.

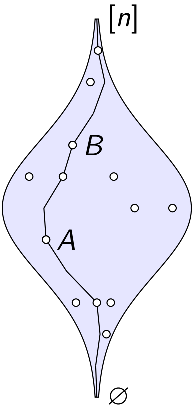
$$\mathbf{E}[\tbinom{X}{2}] = \sum_{A,B} \mathbf{E}\left[Y_{A,B}\right]$$

$$= \sum_{A,B} \frac{1}{\binom{n}{|A|,|B|-|A|,n-|B|}}$$

$$\vdots$$

$$= \sum_{k=1}^{n} \frac{1}{\binom{n}{k}} \sum_{S \in \binom{[n]}{k}} h_{n-k}(\mathcal{F}_S),$$

The Second Moment



- $\binom{X}{2}$ also gives useful information.
- For each ordered pair (A, B) of distinct elements in $\mathcal{F}$ with $A \subsetneq B$, let $Y_{A,B}$ be the indicator r.v. for the full chain containing A and B.

$$\mathbf{E}[\tbinom{X}{2}] = \sum_{A,B} \mathbf{E}\left[Y_{A,B}\right]$$

$$= \sum_{A,B} \frac{1}{\binom{n}{|A|,|B|-|A|,n-|B|}}$$

$$\vdots$$

$$= \sum_{k=1}^{n} \frac{1}{\binom{n}{k}} \sum_{S \in \binom{[n]}{k}} h_{n-k}(\mathcal{F}_S),$$

where $\mathcal{F}_S$ is the set of all $A \in \mathcal{F}$ that are disjoint from S with $A \cup S \in \mathcal{F}$.

Extension of Szemerédi's Cube Lemma

Theorem

If $\mathcal{F} \subseteq 2^{[n]}$ *and* $h_n(\mathcal{F}) > \alpha_d(n)$*, then* $\mathcal{F}$ *contains a copy of* $\mathcal{B}_d$*.*

Extension of Szemerédi's Cube Lemma

Theorem

If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $\mathcal{B}_d$.

Corollary (Szemerédi's Cube Lemma)

If $A \subseteq [0, n]$ and $|A| > \alpha_d(n)$, then A contains an affine d-cube.

Extension of Szemerédi's Cube Lemma

Theorem

If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $\mathcal{B}_d$.

Corollary (Szemerédi's Cube Lemma)

If $A \subseteq [0, n]$ and $|A| > \alpha_d(n)$, then A contains an affine d-cube.

Proof.

- Let $\mathcal{F} = \bigcup_{k \in A} \binom{[n]}{k}$.

□

Extension of Szemerédi's Cube Lemma

Theorem
If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $\mathcal{B}_d$.

Corollary (Szemerédi's Cube Lemma)
If $A \subseteq [0, n]$ and $|A| > \alpha_d(n)$, then A contains an affine d-cube.

Proof.

- Let $\mathcal{F} = \bigcup_{k \in A} \binom{[n]}{k}$.
- Note $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) = |A| > \alpha_d(n)$.

□

Extension of Szemerédi's Cube Lemma

Theorem

If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $\mathcal{B}_d$.

Corollary (Szemerédi's Cube Lemma)

If $A \subseteq [0, n]$ and $|A| > \alpha_d(n)$, then A contains an affine d-cube.

Proof.

- Let $\mathcal{F} = \bigcup_{k \in A} \binom{[n]}{k}$.
- Note $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) = |A| > \alpha_d(n)$.
- By the theorem: $\mathcal{F}$ contains a copy of $\mathcal{B}_d$ generated by disjoint sets $X_0, X_1, \ldots, X_d$.

□

Extension of Szemerédi's Cube Lemma

Theorem

If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $\mathcal{B}_d$.

Corollary (Szemerédi's Cube Lemma)

If $A \subseteq [0, n]$ and $|A| > \alpha_d(n)$, then A contains an affine d-cube.

Proof.

- Let $\mathcal{F} = \bigcup_{k \in A} \binom{[n]}{k}$.
- Note $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) = |A| > \alpha_d(n)$.
- By the theorem: $\mathcal{F}$ contains a copy of $\mathcal{B}_d$ generated by disjoint sets $X_0, X_1, \ldots, X_d$.
- Hence A contains an affine d-cube generated by $x_0, \ldots, x_d$ with $x_i = |X_i|$.

□

Extension of Szemerédi's Cube Lemma

Theorem

If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $\mathcal{B}_d$.

Corollary (Szemerédi's Cube Lemma)

If $A \subseteq [0, n]$ and $|A| > \alpha_d(n)$, then A contains an affine d-cube.

Question

- Is it true that among all $\mathcal{B}_d$-free families $\mathcal{F} \subseteq 2^{[n]}$ that maximize $h_n(\mathcal{F})$, at least one is the union of level sets?

Extension of Szemerédi's Cube Lemma

Theorem

If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $\mathcal{B}_d$.

Corollary (Szemerédi's Cube Lemma)

If $A \subseteq [0, n]$ and $|A| > \alpha_d(n)$, then A contains an affine d-cube.

Question

- Is it true that among all $\mathcal{B}_d$-free families $\mathcal{F} \subseteq 2^{[n]}$ that maximize $h_n(\mathcal{F})$, at least one is the union of level sets?
- If so, then both extremal problems are equivalent.

Extension of Szemerédi's Cube Lemma

Theorem

If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $\mathcal{B}_d$.

Corollary (Szemerédi's Cube Lemma)

If $A \subseteq [0, n]$ and $|A| > \alpha_d(n)$, then A contains an affine d-cube.

Question

- Is it true that among all $\mathcal{B}_d$-free families $\mathcal{F} \subseteq 2^{[n]}$ that maximize $h_n(\mathcal{F})$, at least one is the union of level sets?
- If so, then both extremal problems are equivalent.
- Sperner's Theorem: yes for $d = 1$.

Extension of Szemerédi's Cube Lemma

Theorem
If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $\mathcal{B}_d$.

Corollary (Szemerédi's Cube Lemma)
If $A \subseteq [0, n]$ and $|A| > \alpha_d(n)$, then A contains an affine d-cube.

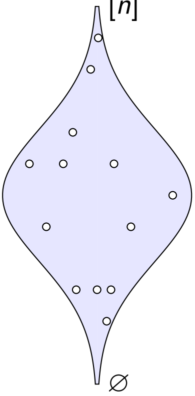
Question

- Is it true that among all $\mathcal{B}_d$-free families $\mathcal{F} \subseteq 2^{[n]}$ that maximize $h_n(\mathcal{F})$, at least one is the union of level sets?
- If so, then both extremal problems are equivalent.
- Sperner's Theorem: yes for $d = 1$.
- Open for $d \geq 2$.

Extension of Szemerédi's Cube Lemma: Proof

Theorem

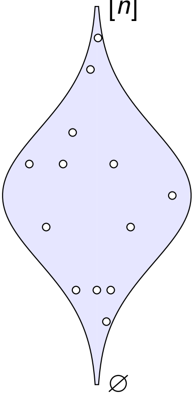
If $\mathcal{F} \subseteq 2^{[n]}$ *and* $h_n(\mathcal{F}) > \alpha_d(n)$, *then* $\mathcal{F}$ *contains a copy of* $\mathcal{B}_d$.



Extension of Szemerédi's Cube Lemma: Proof

Theorem

If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $\mathcal{B}_d$.

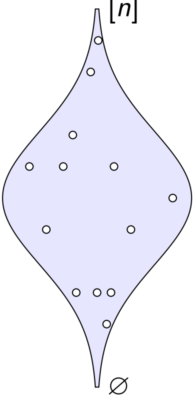


- By induction on d. Case $d = 0$: trivial.

Extension of Szemerédi's Cube Lemma: Proof

Theorem

If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $\mathcal{B}_d$.

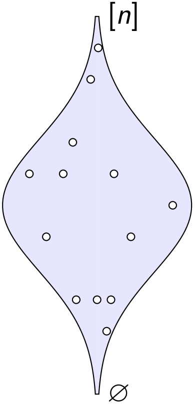


- By induction on d. Case $d = 0$: trivial.
- Let X be the number of times a random full chain meets $\mathcal{F}$.

Extension of Szemerédi's Cube Lemma: Proof

Theorem

If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $\mathcal{B}_d$.

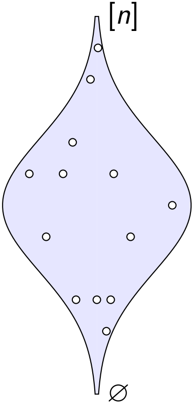


- By induction on d. Case $d = 0$: trivial.
- Let X be the number of times a random full chain meets $\mathcal{F}$.
- $\mathbf{E}[X] = h_n(\mathcal{F}) > \alpha_d(n)$.

Extension of Szemerédi's Cube Lemma: Proof

Theorem

If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $\mathcal{B}_d$.



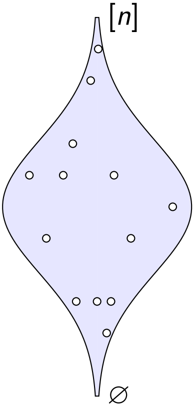
- By induction on d. Case $d = 0$: trivial.
- Let X be the number of times a random full chain meets $\mathcal{F}$.
- $\mathbf{E}[X] = h_n(\mathcal{F}) > \alpha_d(n)$.
- By convexity:

$$\mathbf{E}[\tbinom{X}{2}] \geq \tbinom{\mathbf{E}[X]}{2}$$

Extension of Szemerédi's Cube Lemma: Proof

Theorem

If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $\mathcal{B}_d$.



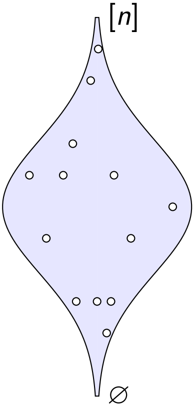
- By induction on d. Case $d = 0$: trivial.
- Let X be the number of times a random full chain meets $\mathcal{F}$.
- $\mathbf{E}[X] = h_n(\mathcal{F}) > \alpha_d(n)$.
- By convexity:

$$\mathbf{E}[\binom{X}{2}] \geq \binom{\mathbf{E}[X]}{2} > \binom{\alpha_d(n)}{2}$$

Extension of Szemerédi's Cube Lemma: Proof

Theorem

If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $\mathcal{B}_d$.



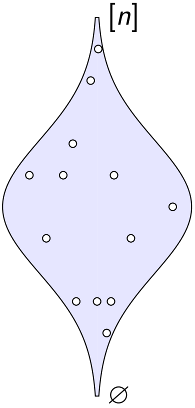
- By induction on d. Case $d = 0$: trivial.
- Let X be the number of times a random full chain meets $\mathcal{F}$.
- $\mathbf{E}[X] = h_n(\mathcal{F}) > \alpha_d(n)$.
- By convexity:

$$\mathbf{E}[\tbinom{X}{2}] \geq \tbinom{\mathbf{E}[X]}{2} > \tbinom{\alpha_d(n)}{2} = n\alpha_{d-1}(n)$$

Extension of Szemerédi's Cube Lemma: Proof

Theorem

If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $\mathcal{B}_d$.



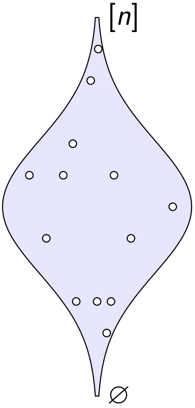
- By convexity:

$$\mathbf{E}[\binom{X}{2}] \geq \binom{\mathbf{E}[X]}{2} > \binom{\alpha_d(n)}{2} = n\alpha_{d-1}(n)$$

Extension of Szemerédi's Cube Lemma: Proof

Theorem

If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $\mathcal{B}_d$.



- By convexity:

$$\mathbf{E}[\tbinom{X}{2}] \geq \tbinom{\mathbf{E}[X]}{2} > \tbinom{\alpha_d(n)}{2} = n\alpha_{d-1}(n)$$

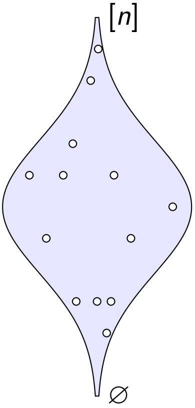
- Grouping pairs $(A, B) \in \mathcal{F} \times \mathcal{F}$ with $A \subsetneq B$ by $B - A$, with $S = B - A$:

$$\mathbf{E}[\tbinom{X}{2}] =$$

Extension of Szemerédi's Cube Lemma: Proof

Theorem

If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $\mathcal{B}_d$.

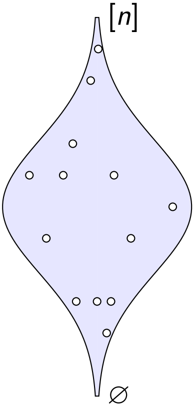


- By convexity:
$$\mathbf{E}[\tbinom{X}{2}] \geq \tbinom{\mathbf{E}[X]}{2} > \tbinom{\alpha_d(n)}{2} = n\alpha_{d-1}(n)$$
- Grouping pairs $(A, B) \in \mathcal{F} \times \mathcal{F}$ with $A \subsetneq B$ by $B - A$, with $S = B - A$:
$$\mathbf{E}[\tbinom{X}{2}] = \sum_{k=1}^{n} \frac{1}{\binom{n}{k}} \sum_{S \in \binom{[n]}{k}} h_{n-k}(\mathcal{F}_S),$$

Extension of Szemerédi's Cube Lemma: Proof

Theorem

If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $\mathcal{B}_d$.



- By convexity:

$$\mathbf{E}[\tbinom{X}{2}] \geq \binom{\mathbf{E}[X]}{2} > \binom{\alpha_d(n)}{2} = n\alpha_{d-1}(n)$$

- Grouping pairs $(A, B) \in \mathcal{F} \times \mathcal{F}$ with $A \subsetneq B$ by $B - A$, with $S = B - A$:

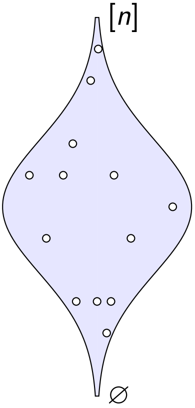
$$\mathbf{E}[\tbinom{X}{2}] = \sum_{k=1}^{n} \frac{1}{\binom{n}{k}} \sum_{S \in \binom{[n]}{k}} h_{n-k}(\mathcal{F}_S),$$

where $\mathcal{F}_S$ is the family of all $A \in \mathcal{F}$ that are disjoint from S with $A \cup S \in \mathcal{F}$.

Extension of Szemerédi's Cube Lemma: Proof

Theorem

If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $\mathcal{B}_d$.



- By convexity:

$$\mathbf{E}\left[\binom{X}{2}\right] \geq \binom{\mathbf{E}[X]}{2} > \binom{\alpha_d(n)}{2} = n\alpha_{d-1}(n)$$

- Grouping pairs $(A, B) \in \mathcal{F} \times \mathcal{F}$ with $A \subsetneq B$ by $B - A$, with $S = B - A$:

$$\mathbf{E}\left[\binom{X}{2}\right] = \sum_{k=1}^{n} \frac{1}{\binom{n}{k}} \sum_{S \in \binom{[n]}{k}} h_{n-k}(\mathcal{F}_S),$$

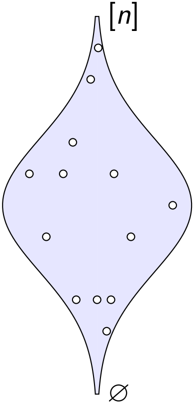
  where $\mathcal{F}_S$ is the family of all $A \in \mathcal{F}$ that are disjoint from S with $A \cup S \in \mathcal{F}$.

- $\displaystyle\sum_{k=1}^{n} \frac{1}{\binom{n}{k}} \sum_{S \in \binom{[n]}{k}} h_{n-k}(\mathcal{F}_S) > n\alpha_{d-1}(n)$

Extension of Szemerédi's Cube Lemma: Proof

Theorem

If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $\mathcal{B}_d$.

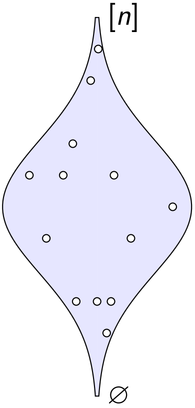


- $$\sum_{k=1}^{n} \frac{1}{\binom{n}{k}} \sum_{S \in \binom{[n]}{k}} h_{n-k}(\mathcal{F}_S) > n\alpha_{d-1}(n)$$

Extension of Szemerédi's Cube Lemma: Proof

Theorem

If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $\mathcal{B}_d$.

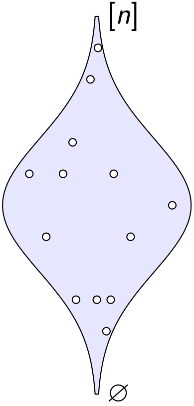


- $$\sum_{k=1}^{n} \frac{1}{\binom{n}{k}} \sum_{S \in \binom{[n]}{k}} h_{n-k}(\mathcal{F}_S) > n\alpha_{d-1}(n)$$
- Find k such that
$$\frac{1}{\binom{n}{k}} \sum_{S \in \binom{[n]}{k}} h_{n-k}(\mathcal{F}_S) > \alpha_{d-1}(n)$$

Extension of Szemerédi's Cube Lemma: Proof

Theorem

If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $\mathcal{B}_d$.

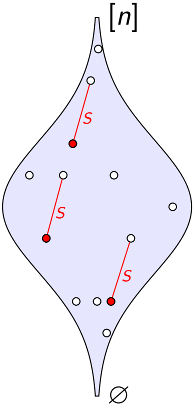


- $\displaystyle\sum_{k=1}^{n} \frac{1}{\binom{n}{k}} \sum_{S \in \binom{[n]}{k}} h_{n-k}(\mathcal{F}_S) > n\alpha_{d-1}(n)$
- Find k such that
$$\frac{1}{\binom{n}{k}} \sum_{S \in \binom{[n]}{k}} h_{n-k}(\mathcal{F}_S) > \alpha_{d-1}(n)$$
- Find $S \in \binom{[n]}{k}$ with $h_{n-k}(\mathcal{F}_S) > \alpha_{d-1}(n)$.

Extension of Szemerédi's Cube Lemma: Proof

Theorem

If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $\mathcal{B}_d$.

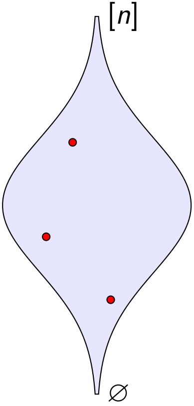


- $$\sum_{k=1}^{n} \frac{1}{\binom{n}{k}} \sum_{S \in \binom{[n]}{k}} h_{n-k}(\mathcal{F}_S) > n\alpha_{d-1}(n)$$
- Find k such that
$$\frac{1}{\binom{n}{k}} \sum_{S \in \binom{[n]}{k}} h_{n-k}(\mathcal{F}_S) > \alpha_{d-1}(n)$$
- Find $S \in \binom{[n]}{k}$ with $h_{n-k}(\mathcal{F}_S) > \alpha_{d-1}(n)$.

Extension of Szemerédi's Cube Lemma: Proof

Theorem

If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $\mathcal{B}_d$.

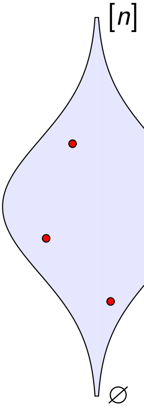


- $$\sum_{k=1}^{n} \frac{1}{\binom{n}{k}} \sum_{S \in \binom{[n]}{k}} h_{n-k}(\mathcal{F}_S) > n\alpha_{d-1}(n)$$
- Find k such that
$$\frac{1}{\binom{n}{k}} \sum_{S \in \binom{[n]}{k}} h_{n-k}(\mathcal{F}_S) > \alpha_{d-1}(n)$$
- Find $S \in \binom{[n]}{k}$ with $h_{n-k}(\mathcal{F}_S) > \alpha_{d-1}(n)$.

Extension of Szemerédi's Cube Lemma: Proof

Theorem

If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $\mathcal{B}_d$.

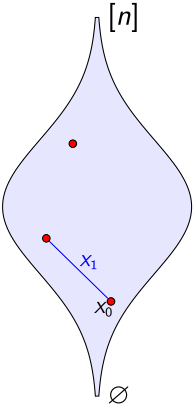


- $$\sum_{k=1}^{n} \frac{1}{\binom{n}{k}} \sum_{S \in \binom{[n]}{k}} h_{n-k}(\mathcal{F}_S) > n\alpha_{d-1}(n)$$
- Find k such that
  $$\frac{1}{\binom{n}{k}} \sum_{S \in \binom{[n]}{k}} h_{n-k}(\mathcal{F}_S) > \alpha_{d-1}(n)$$
- Find $S \in \binom{[n]}{k}$ with $h_{n-k}(\mathcal{F}_S) > \alpha_{d-1}(n)$.
- By induction, $\mathcal{F}_S$ contains a copy of $\mathcal{B}_{d-1}$ generated by $X_0, \ldots, X_{d-1}$.

Extension of Szemerédi's Cube Lemma: Proof

Theorem

If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $\mathcal{B}_d$.

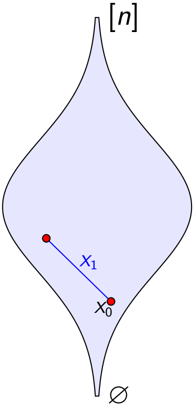


- $$\sum_{k=1}^{n} \frac{1}{\binom{n}{k}} \sum_{S \in \binom{[n]}{k}} h_{n-k}(\mathcal{F}_S) > n\alpha_{d-1}(n)$$
- Find k such that
$$\frac{1}{\binom{n}{k}} \sum_{S \in \binom{[n]}{k}} h_{n-k}(\mathcal{F}_S) > \alpha_{d-1}(n)$$
- Find $S \in \binom{[n]}{k}$ with $h_{n-k}(\mathcal{F}_S) > \alpha_{d-1}(n)$.
- By induction, $\mathcal{F}_S$ contains a copy of $\mathcal{B}_{d-1}$ generated by $X_0, \ldots, X_{d-1}$.

Extension of Szemerédi's Cube Lemma: Proof

Theorem

If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $\mathcal{B}_d$.

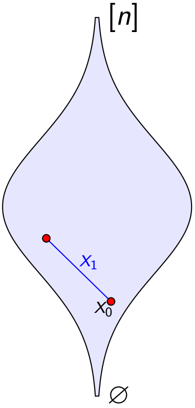


- $$\sum_{k=1}^{n} \frac{1}{\binom{n}{k}} \sum_{S \in \binom{[n]}{k}} h_{n-k}(\mathcal{F}_S) > n\alpha_{d-1}(n)$$
- Find k such that
$$\frac{1}{\binom{n}{k}} \sum_{S \in \binom{[n]}{k}} h_{n-k}(\mathcal{F}_S) > \alpha_{d-1}(n)$$
- Find $S \in \binom{[n]}{k}$ with $h_{n-k}(\mathcal{F}_S) > \alpha_{d-1}(n)$.
- By induction, $\mathcal{F}_S$ contains a copy of $\mathcal{B}_{d-1}$ generated by $X_0, \ldots, X_{d-1}$.

Extension of Szemerédi's Cube Lemma: Proof

Theorem

If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $\mathcal{B}_d$.

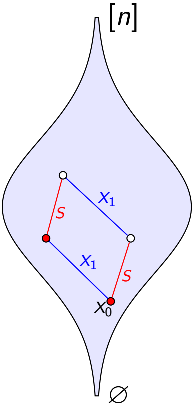


- $$\sum_{k=1}^{n} \frac{1}{\binom{n}{k}} \sum_{S \in \binom{[n]}{k}} h_{n-k}(\mathcal{F}_S) > n\alpha_{d-1}(n)$$
- Find k such that
$$\frac{1}{\binom{n}{k}} \sum_{S \in \binom{[n]}{k}} h_{n-k}(\mathcal{F}_S) > \alpha_{d-1}(n)$$
- Find $S \in \binom{[n]}{k}$ with $h_{n-k}(\mathcal{F}_S) > \alpha_{d-1}(n)$.
- By induction, $\mathcal{F}_S$ contains a copy of $\mathcal{B}_{d-1}$ generated by $X_0, \ldots, X_{d-1}$.
- $\mathcal{F}$ contains a copy of $\mathcal{B}_d$ generated by $X_0, \ldots, X_d$ with $X_d = S$.

Extension of Szemerédi's Cube Lemma: Proof

Theorem

If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $\mathcal{B}_d$.



- $$\sum_{k=1}^{n} \frac{1}{\binom{n}{k}} \sum_{S \in \binom{[n]}{k}} h_{n-k}(\mathcal{F}_S) > n\alpha_{d-1}(n)$$
- Find k such that
$$\frac{1}{\binom{n}{k}} \sum_{S \in \binom{[n]}{k}} h_{n-k}(\mathcal{F}_S) > \alpha_{d-1}(n)$$
- Find $S \in \binom{[n]}{k}$ with $h_{n-k}(\mathcal{F}_S) > \alpha_{d-1}(n)$.
- By induction, $\mathcal{F}_S$ contains a copy of $\mathcal{B}_{d-1}$ generated by $X_0, \ldots, X_{d-1}$.
- $\mathcal{F}$ contains a copy of $\mathcal{B}_d$ generated by $X_0, \ldots, X_d$ with $X_d = S$.

Turán Results

Theorem

If $\mathcal{F} \subseteq 2^{[n]}$ *and* $h_n(\mathcal{F}) > \alpha_d(n)$*, then* $\mathcal{F}$ *contains a copy of* $\mathcal{B}_d$*.*

Turán Results

Theorem
If $\mathcal{F} \subseteq 2^{[n]}$ *and* $h_n(\mathcal{F}) > \alpha_d(n)$, *then* $\mathcal{F}$ *contains a copy of* $\mathcal{B}_d$.

Corollary
If $\mathcal{F} \subseteq 2^{[n]}$ *and* $h_n(\mathcal{F}) \geq 4n^{1-\frac{2}{2^d}}$, *then* $\mathcal{F}$ *contains a copy of* $\mathcal{B}_d$.

Turán Results

Theorem

If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $\mathcal{B}_d$.

Corollary

If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) \geq 4n^{1-\frac{2}{2^d}}$, then $\mathcal{F}$ contains a copy of $\mathcal{B}_d$.

- Partitioning $2^{[n]}$ into consecutive segments of $\sqrt{n}$ levels and applying an averaging argument yields:

Turán Results

Theorem
If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a copy of $\mathcal{B}_d$.

Corollary
If $\mathcal{F} \subseteq 2^{[n]}$ and $h_n(\mathcal{F}) \geq 4n^{1-\frac{2}{2^d}}$, then $\mathcal{F}$ contains a copy of $\mathcal{B}_d$.

- Partitioning $2^{[n]}$ into consecutive segments of $\sqrt{n}$ levels and applying an averaging argument yields:

Theorem
If $\mathcal{F} \subseteq 2^{[n]}$ and $|\mathcal{F}| \geq 50n^{-1/2^d} \cdot 2^n$, then $\mathcal{F}$ contains a copy of $\mathcal{B}_d$.

Ramsey Problem

- How many parts are needed to partition $2^{[n]}$ into $\mathcal{B}_d$-free families?

Ramsey Problem

- How many parts are needed to partition $2^{[n]}$ into $\mathcal{B}_d$-free families?
- Let $r(n, d)$ be the minimum number of parts needed.

Ramsey Problem

- How many parts are needed to partition $2^{[n]}$ into $\mathcal{B}_d$-free families?
- Let $r(n, d)$ be the minimum number of parts needed.

Prior Work

- Clearly, $r(n, 1) = n + 1$.

Ramsey Problem

- How many parts are needed to partition $2^{[n]}$ into $\mathcal{B}_d$-free families?
- Let $r(n, d)$ be the minimum number of parts needed.

Prior Work

- Clearly, $r(n, 1) = n + 1$.
- [Gunderson–Rödl–Sidorenko 1999] For n sufficiently large

$$(1 - o(1))\frac{3}{4} \cdot n^{1/2} \leq r(n, 2) \leq (1 + o(1)) \cdot n^{1/2}.$$

Ramsey Problem

- How many parts are needed to partition $2^{[n]}$ into $\mathcal{B}_d$-free families?
- Let $r(n, d)$ be the minimum number of parts needed.

Prior Work

- [Gunderson–Rödl–Sidorenko 1999] For $d > 2$, there exists c_d such that for n sufficiently large

$$c_d \cdot n^{\frac{1}{2^d}} \leq r(n, d) \leq n^{\frac{d}{2^d - 1}(1+o(1))}.$$

Ramsey Problem

- How many parts are needed to partition $2^{[n]}$ into $\mathcal{B}_d$-free families?
- Let $r(n, d)$ be the minimum number of parts needed.

Prior Work

- [Gunderson–Rödl–Sidorenko 1999] For $d > 2$, there exists c_d such that for n sufficiently large

$$c_d \cdot n^{\frac{1}{2^d}} \leq r(n, d) \leq n^{\frac{d}{2^d - 1}(1+o(1))}.$$

- Here, $c_d = (10d)^{-d}(1 + o(1))$.

Ramsey Problem

- How many parts are needed to partition $2^{[n]}$ into $\mathcal{B}_d$-free families?
- Let $r(n,d)$ be the minimum number of parts needed.

Prior Work

- [Gunderson–Rödl–Sidorenko 1999] For $d > 2$, there exists c_d such that for n sufficiently large

$$c_d \cdot n^{\frac{1}{2^d}} \leq r(n,d) \leq n^{\frac{d}{2^d-1}(1+o(1))}.$$

- Here, $c_d = (10d)^{-d}(1+o(1))$.

Theorem

$$r(n,d) \geq \frac{1}{4} \cdot n^{\frac{2}{2^d}}$$

Ramsey Problem

- How many parts are needed to partition $2^{[n]}$ into $\mathcal{B}_d$-free families?
- Let $r(n, d)$ be the minimum number of parts needed.

Prior Work

- [Gunderson–Rödl–Sidorenko 1999] For $d > 2$, there exists c_d such that for n sufficiently large

$$c_d \cdot n^{\frac{1}{2^d}} \leq r(n, d) \leq n^{\frac{d}{2^d - 1}(1+o(1))}.$$

- Here, $c_d = (10d)^{-d}(1 + o(1))$.

Theorem

$$r(n, d) \geq \frac{1}{4} \cdot n^{\frac{2}{2^d}}$$

Thank You.